Baku Mathematical Journal
2022, Vol. 1, No 2, PP. 195-204
https://doi.org/10.32010/j.bmj.2022.20

INVERSE BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFUSION EQUATION WITH NON-LOCAL TIME-INTEGRAL CONDITIONS OF THE SECOND KIND

M.Yu. SALIMOV*, **A.A. SALIMOVA**

Received: 30.03.2022 / Revised: 20.07.2022 / Accepted: 27.07.2022

Abstract. *This work deals with the solvability of inverse boundary value problem with time-dependent unknown coefficient for nonlinear diffusion equation. Definition of classical solution for the considered inverse boundary value problem is introduced. By the Fourier method, the problem is reduced to the system of integral equations. Using the contraction mappings method, the existence and uniqueness of the solution of the system of integral equations are proved. Finally, the existence and uniqueness of the classical solution of original problem are proved.*

Keywords: inverse boundary value problem, nonlinear diffusion equation, Fourier method, classical solution

Mathematics Subject Classification (2020): 35R30, 35L10, 35A01, 35A02, 35A09

1. Introduction

By the inverse boundary value problem for partial differential equations, we mean in this work a problem which requires finding not only the solution, but also the right-hand side and (or) some coefficient (coefficients) of the equation. Inverse problems occur in various fields of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc which puts them in a number of urgent problems of modern mathematics. In case where the unknowns in the inverse problem are the solution and the right-hand side, then this kind of inverse problem is linear; and in case where

* Corresponding author.

Matlab Yu. Salimov
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: m.y.salimov@gmail.com, matlab.salimov@asoiu.edu.az

Afarim A. Salimova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: aferims@mail.ru

the unknowns are the solution and at least one of the coefficients, then such an inverse problem is nonlinear.

One of the "reaction-diffusion" type nonlinear equations is a logistic equation with diffusion

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + ku(x,t)(1-u(x,t)). \tag{1}$$

This equation was introduced by A. N. Kolmogorov, I. G. Petrovski and N. S. Piskunov [8] and Fisher [2] to model the process of gene wave propagation. The equation (1), also called Fisher-Kolmogorov-Petrovski-Piskunov equation (FKPP), has wide applications in many fields such as heat and mass transfer problems, combustion theory, biology, ecology, plasma physics, theory of phase transitions, etc [2], [8].

It should be noted that many characteristics of physical, chemical, biological, ecological, etc processes described by the equation (1) largely depend on the coefficients of the given equation. Therefore, the problems of finding these coefficients to provide best conditions for the course of processes are important.

In this work, the existence and uniqueness of the solution of nonlocal inverse boundary value problem for nonlinear diffusion equation are proved by using the Fourier method and the contraction mapping principle.

Inverse boundary value problems for a second order parabolic equation have been considered in [1], [3]–[6], [9].

2. Statement of Inverse Boundary Value Problem

Consider the inverse boundary value problem of finding the solution and the unknown coefficient of the parabolic equation

$$a(t)u_t(x,t) = u_{xx}(x,t) + p(t)u(x,t)(1-u(x,t)) + f(x,t),$$
$$D_T = \{(x,t) : 0 \le x \le 1, \quad 0 \le t \le T\}, \tag{2}$$

with the conditions

$$u(x,0) + \int_0^T b(t)u(x,t)dt = \varphi(x) \quad (0 \le x \le 1)\,, \tag{3}$$

$$u(0,t) = 0\,, \quad u_x(1,t) = 0 \quad (0 \le t \le T), \tag{4}$$

$$u(1,t) = h(t) \quad (0 \le t \le T), \tag{5}$$

where $a(t) > 0$, $b(t)$,$f(x,t)$, $\varphi(x)$, $h(t)$ are the given functions, and $u(x,t)$ and $p(t)$ are the sought functions.

Definition. *We will call a pair $\{u(x,t), p(t)\}$ of functions $u(x,t)$ and $p(t)$ a classical solution of the problem (2)-(5), if the following conditions are satisfied:*

1) the function $u(x,t)$ and its derivatives $u_t(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ are continuous in D_T;

2) the function $p(t)$ is continuous in $[0,T]$;

3) the equation (2) and the conditions (3)-(5) are satisfied in the usual classical sense.

The following theorem is true.

Theorem 1. *Let* $0 \le b(t) \in C[0,T]$*, the function* $a(t)$ *be positive and continuous in* $[0,T]$*, the function* $\varphi(x)$ *be continuous in* $[0,1]$*, and the function* $f(x,t)$ *be continuous in totality of variables in* D_T*,*$h(t) \in C^1[0,T]$*,* $h(t)(1-h(t)) \neq 0$ $(0 \le t \le T)$*. Moreover, let the condition of coordination*

$$\varphi(1) = h(0) + \int_0^T b(t)h(t)dt \tag{6}$$

holds. Then the problem of finding classical solution of the problem (2)-(5) is equivalent to the problem of finding the functions $u(x,t) \in C^{2,1}(D_T)$ *and* $p(t) \in C[0,T]$ *from the relations (2)-(4) such that*

$$a(t)h'(t) = u_{xx}(1,t) + p(t)h(t)(1-h(t)) + f(1,t) \quad (0 \le t \le T). \tag{7}$$

Proof. Let $\{u(x,t), p(t)\}$ be a classical solution of the problem (2)-(5). Assuming that $h(t)$ is differentiable, from (5) we get:

$$u_t(1,t) = h'(t) \ \ (0 \le t \le T). \tag{8}$$

Substituting $x = 1$ into the equation (2), we have

$$a(t)\, u_t(1,t) = u_{xx}(1,t) + p(t)u(1,t)(1-u(1,t)) + f(1,t) \quad (0 \le t \le T). \tag{9}$$

From the last relation, by (5) and (8), it follows that (7) holds.

Now let $\{u(x,t), p(t)\}$ be a solution of the problem (2)-(4), (7) and let the condition of coordination (6) be satisfied. From (7) and (9) we have

$$a(t)\frac{d}{dt}(u(1,t)-h(t)) = p(t)(u(1,t)-h(t))\,(1-(u(1,t)+h(t)) \quad (0 \le t \le T). \tag{10}$$

From (10), by (3) and (6), it is not difficult to see that

$$u(1,0) - h(0) + \int_0^T b(t)(u(x,t)-h(t))dt = u(1,0) + \int_0^T b(t)u(x,t)dt -$$

$$-(h(0) + \int_0^T b(t)h(t)dt) = \varphi(1) - (h(0) + \int_0^T b(t)h(t)dt) = 0. \tag{11}$$

Obviously, the general solution of (10) has the form

$$u(1,t) - h(t) = ce^{\int_0^t \frac{p(\tau)(1-(u(1,\tau)+h(\tau)))}{a(\tau)}d\tau} \quad (0 \le t \le T). \tag{12}$$

Hence, by (11), we obtain

$$c\left(1 + \int_0^T b(t)e^{\int_0^t \frac{p(\tau)(1-(u(0,\tau)+h(\tau)))}{a(\tau)}d\tau}dt\right) = 0. \tag{13}$$

Due to $b(t) \ge 0$, from (13) we obtain $c = 0$. Substituting this into (12), we conclude that $u(1,t) - h(t) = 0$ $(0 \le t \le T)$. Consequently, it is clear that the condition (5) also holds. The theorem is proved. ◀

3. On Solvability of Inverse Boundary Value Problem

Formally searching for the first component $u(x,t)$ of the solution $\{u(x,t),p(t)\}$ of the problem (2)-(4), (7) in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x \quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{14}$$

we arrive at the problem

$$a(t)u_k'(t)+\lambda_k^2u_k(t)=F_k(t;p,u) \quad (k=1,2,...;0\le t\le T), \tag{15}$$

$$u_k(0)+\int_0^T b(t)u_k(t)dt=\varphi_k \quad (k=1,2,...), \tag{16}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k xdx, \quad F_k(t;u,p)=f_k(t)+p(t)g_k(u)(t),$$

$$g_k(u)(t)=2\int_0^1 u(x,t)(1-u(x,t))\sin\lambda_k xdx, \quad f_k(t)=2\int_0^1 f(x,t)\sin\lambda_k x,$$

$$\varphi_k=\int_0^1\varphi(x)\sin\lambda_k xdx \quad (k=1,2,...).$$

Further, from (15), (16) we obtain

$$u_k(t)=\left(\varphi_k-\int_0^T b(t)u_k(t)dt\right)e^{-\int_0^t\frac{\lambda_k^2ds}{a(s)}}+\int_0^t\frac{F_k(\tau;p,u)}{a(\tau)}e^{-\int_\tau^t\frac{\lambda_k^2ds}{a(s)}}d\tau \quad (k=1,2,...). \tag{17}$$

To find the first component $u(x,t)$ of the solution $\{u(x,t),p(t)\}$ of the problem (2)-(4), (7), we substitute the expressions $u_k(t)$ $(k=1,2,\ldots)$ from (17) into (14):

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\left(\varphi_k-\int_0^T b(t)u_k(t)dt\right)e^{-\int_0^t\frac{\lambda_k^2ds}{a(s)}}+\right.$$

$$\left.+\int_0^t\frac{F_k(\tau;p,u)}{a(\tau)}e^{-\int_\tau^t\frac{\lambda_k^2ds}{a(s)}}d\tau\right\}\cos\lambda_x x. \tag{18}$$

Taking into account (14), from (7) we obtain

$$p(t)=[h(t)(1-h(t))]^{-1}\left\{a(t)h'(t)-f(1,t)-\sum_{k=1}^{\infty}\lambda_k^2(-1)^ku_k(t)\right\}. \tag{19}$$

Further, substituting the expressions $u_k(t)$ from (17) into (19), we obtain the following relation for the second component $p(t)$ of the solution $\{u(x,t),p(t)\}$ of the problem (2)-(4), (7):

$$p(t)=[h(t)(1-h(t))]^{-1}\left\{a(t)h'(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2(-1)^k\times\right.$$

$$\times\left[\left(\varphi_k-\int_0^T b(t)u_k(t)dt\right)e^{-\int_0^t\frac{\lambda_k^2 ds}{a(s)}}+\int_0^t\frac{F_k(\tau;p,u)}{a(\tau)}e^{-\int_\tau^t\frac{\lambda_k^2 ds}{a(s)}}d\tau\right]\right\}. \qquad (20)$$

Thus, the solution of the problem (2)-(4), (7) is reduced to the solution of the system (18) and (20) with respect to the unknown functions $u(x,t)$ and $p(t)$.

The lemma below is important for the uniqueness of the solution of the problem (2)-(4), (7).

Lemma. *If $\{u(x,t),p(t)\}$ is a solution of the problem (2)-(4), (7), then the functions*

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k=1,2,...)$$

satisfy the countable system (17) on $[0,T]$.

Proof. Let $\{u(x,t),p(t)\}$ be any solution of (2)-(4), (7). Then, by multiplying both sides of the equation (2) by the function $2\sin\lambda_k x$ $(k=1,2,...)$, integrating the obtained equality with respect to x from 0 to 1 and using the relations

$$2\int_0^1 u_t(x,t)\sin\lambda_k x dx=\frac{d}{dt}\left(2\int_o^1 u(x,t)\sin\lambda_k x dx\right)=u_k'(t)(k=1,2,...),$$

$$\int_0^1 u_{xx}(x,t)\sin\lambda_k x dx=-\lambda_k^2\left(2\int_0^1 u(x,t)\sin\lambda_k x dx\right)=-\lambda_k^2 u_k(t),$$

we see that the relations (15) are satisfied.

Similarly, from (3) it follows that the condition (16) is satisfied.

Thus, $u_k(t)$ $(k=0,1,...)$ is a solution of the problem (15), (16). Hence it immediately follows that the functions $u_k(t)$ $(k=0,1,...)$ satisfy the system (17) on $[0,T]$. The lemma is proved. ◀

Obviously, if $u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx$ $(k=1,2,...)$ is a solution of the system (17), then the pair $\{u(x,t),p(t)\}$ consisting of the functions $u(x,t)=\sum\limits_{k=1}^{\infty}u_k(t)\sin\lambda_k x$ $\left(\lambda_k=\frac{\pi}{2}(2k-1)\right)$ and $p(t)$ is a solution of the system (18), (20).

Lemma has the following

Corollary. *Let the system (18), (20) have a unique solution. Then the problem (2)-(4), (7) cannot have more than one solution, i.e. if the problem (2)-(4), (7) has a solution, then it is unique.*

Now, to study the problem (2)-(4), (7), consider the following spaces.

1. Denote by $B_{2,T}^3$ [7] a totality of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x \quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right)$$

considered in D_T, where each of the functions $u_k(t)$ is continuous in $[0,T]$ and

$$J_T(u) \equiv \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \left\| u_k(t) \right\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} < +\infty.$$

Define the norm in this set as follows:

$$\left\| u(x,t) \right\|_{B_{2,T}^3} = J_T(u).$$

2. By E_T^3 we denote a space defined by the topological product

$$B_{2,T}^3 \times C[0,T].$$

The norm of the element $z = \{u, p\}$ in this space is defined by the formula

$$\left\| z \right\|_{E_T^3} = \left\| u(x,t) \right\|_{B_{2,T}^3} + \left\| p(t) \right\|_{C[0,T]}.$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

Now let's consider in the space E_T^3 the operator

$$\Phi(u,a) = \left\{ \Phi_1(u,p), \Phi_2(u,p) \right\},$$

where

$$\Phi_1(u,p) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t) \sin \lambda_k x, \quad \Phi_2(u,p) = \tilde{p}(t),$$

and $\tilde{u}_k(t)$ $(k = 1, 2, ...)$ and $\tilde{p}(t)$ are equal to the right-hand sides of (17) and (20), respectively.

It is not difficult to see that

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left\| \tilde{u}_k(t) \right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \leq 2 \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left| \varphi_k \right| \right)^2 \right)^{\frac{1}{2}} +$$

$$+2 \left\| b(t) \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left\| u_k(t) \right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + 2\sqrt{T} \left\| \frac{1}{a(t)} \right\|_{C[0,T]} \times$$

$$\times \left[\left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 \left| f_k(\tau) \right| \right)^2 d\tau \right) + \left\| p(t) \right\|_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 \left| g_k(u)(\tau) \right| \right)^2 d\tau \right) \right], \qquad (21)$$

$$\left\| \tilde{p}(t) \right\|_{C[0,T]} \leq \left\| [h(t)(1-h(t))]^{-1} \right\|_{C[0,T]} \left\{ \left\| a(t)h'(t) - f(1,t) \right\|_{C[0,T]} + \right.$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\left(\sum_{k=1}^{\infty} (\lambda_k^3 \left| \varphi_k \right|)^2 \right)^{\frac{1}{2}} + \left\| b(t) \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left\| u_k(t) \right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \right.$$

$$+\sqrt{T}\left\|\frac{1}{a(t)}\right\|_{C[0,T]}\left[\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+\ \|p(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|g_k(u)(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right]\right]\Bigg\}. \qquad (22)$$

Assume that the data of the problem (2)-(4), (7) satisfy the following conditions:
1. $\varphi(x)\in C^2[0,1]$, $\varphi'''(x)\in L_2(0,1)$, $\varphi(0)=\varphi'(1)=\varphi''(0)=0$;
2. $f(x,t), f_x(x,t), f_{xx}(x,t)\in C(D_T)$, $f_{xxx}(x,t)\in L_2(D_T)$;
3. $f(0,t)=f_x(1,t)=f_{xx}(0,t)=0$ $(0\le t\le T)$;
4. $b(t)\in C[0,T]$, $0<a(t)\in C[0,T]$, $h(t)\in C^1[0,T]$, $h(t)(1-h(t))\neq 0$ $(0\le t\le T)$.

Then from (21) and (22), respectively, we have

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\tilde{u}_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\le 2\left\|\varphi'''(x)\right\|_{L_2(0,1)}+$$

$$+2\left\|b(t)\right\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+2\sqrt{T}(1+\delta)\left\|\frac{1}{a(t)}\right\|_{C[0,T]}\times$$

$$\times\left[\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+\|p(t)\|_{C[0,T]}\left\|\frac{\partial^3}{\partial x^3}\left[u(x,t)(1-u(x,t))\right]\right\|_{L_2(D_T)}\right], \qquad (23)$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le\left\|\left[h(t)(1-h(t))\right]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|a(t)h'(t)-f(0,t)\right\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\|b(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\right.$$

$$+\sqrt{T}\left\|\frac{1}{a(t)}\right\|_{C[0,T]}\left[\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+\right.$$

$$\left.\left.+\ \|p(t)\|_{C[0,T]}\left\|\frac{\partial^3}{\partial x^3}\left[u(x,t)(1-u(x,t))\right]\right\|_{L_2(D_T)}\right]\right]\Bigg\}. \qquad (24)$$

It is easy to see that

$$\left\|\frac{\partial^3}{\partial x^3}\left[u(x,t)(1-u(x,t))\right]\right\|_{L_2(D_T)}=$$

$$=\left\|u_{xxx}(x,t)-2u(x,t)u_{xxx}(x,t)-6u_x(x,t)u_{xx}(x,t)\right\|_{L_2(D_T)}\le$$

$$\le\sqrt{3}\left\|u_{xxx}(x,t)\right\|_{L_2(D_T)}+2\sqrt{3}\left\|u(x,t)\right\|_{C(D_T)}\left\|u_{xxx}(x,t)\right\|_{L_2(D_T)}+$$

$$+6\sqrt{3}\left\|u_x(x,t)\right\|_{C(D_T)}\left\|u_{xx}(x,t)\right\|_{L_2(D_T)}\le$$

$$\leq \sqrt{3T}\,\|u(x,t)\|_{B^3_{2,T}} + 8\sqrt{3T}\,\|u(x,t)\|^2_{B^3_{2,T}}\,. \tag{25}$$

Further, by (25), from (23) and (24), respectively, we obtain

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T)+$$

$$+B_1(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\left(1+\|u(x,t)\|_{B^3_{2,T}}\right)+C_1(T)\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{26}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq A_2(T)+$$

$$+B_2(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\left(1+\|u(x,t)\|_{B^3_{2,T}}\right)+C_2(T)\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{27}$$

where

$$A_1(T)=\sqrt{3}\,\|\varphi'''(x)\|_{L_2(0,1)}+\sqrt{3T}\quad\left\|\frac{1}{a(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T)=24T\left\|\frac{1}{a(t)}\right\|_{C[0,T]}.C_1(T)=2\,\|b(t)\|_{C[0,T]}\,T,$$

$$A_2(T)=\left\|[h(t)(1-h(t))]^{-1}\right\|_{C[0,T]}\left\{\|a(t)h'(t)-f(0,t)\|_{C[0,T]}+\right.$$

$$\left.+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\|\varphi'''(x)\|_{L_2(0,1)}+\sqrt{T}\,\left\|\frac{1}{a(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T)=8\sqrt{3}\left\|[h(t)(1-h(t))]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\frac{1}{a(t)}\right\|_{C[0,T]}T\,,$$

$$C_2(T)=\left\|[h(t)(1-h(t))]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|b(t)\|_{C[0,T]}\,T.$$

From the inequalities (26), (27) we conclude

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}}+\|\tilde{p}(t)\|_{C[0,T]}\leq$$

$$\leq A(T)+B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\left(1+\|u(x,t)\|_{B^3_{2,T}}\right)+C(T)\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{28}$$

where

$$A(T)=A_1(T)+A_2(T),\quad B(T)=B_1(T)+B_2(T),C(T)=C_1(T)+C_2(T).$$

So we can prove the following theorem.

Theorem 2. *Assume that the conditions 1.-4. and the inequality*

$$(B(T)(A(T)+2)(A(T)+3)+C(T))(A(T)+2)<1 \tag{29}$$

holds. Then the problem (2)-(4), (7) *has a unique solution in the ball* $K=K_R(\|z\|_{E_T^3}\leq R=A(T)+2)$ *of* E_T^3.

Proof. Let's rewrite the system of equations (18), (20) in the form

$$z=\Phi z, \tag{30}$$

where $z=\{u,p\}$, $\Phi z=\{\Phi_1 z,\Phi_2 z\}$, and $\Phi_i(u,p)(i=1,2)$ are defined by the right-hand sides of the equations (18), (20), respectively.

Consider the operator $\Phi(u,a_0)$ in the ball $K=K_R(\|z\|_{E_T^3}\leq R=A(T)+2)$ of the space E_T^3.

Similar to (28), we obtain the following estimates for arbitrary $z,z_1,z_2\in K_R$:

$$\|\Phi z\|_{E_T^3}\leq A(T)+B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}\left(1+\|u(x,t)\|_{B_{2,T}^3}\right)+$$

$$+C(T)\|u(x,t)\|_{B_{2,T}^3}\leq A(T)+B(T)(A(T)+2)^2(A(T)+3)+C(T)(A(T)+2), \tag{31}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^3}\leq B(T)R(2R+1)\left(\|p_1(t)-p_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}\right)+$$

$$+C(T)\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}. \tag{32}$$

From the inequalities (31) and (32), by (29), it follows that Φ is a contraction operator in the ball $K=K_R$. Consequently, the operator Φ has a unique fixed point $\{u,p\}$ in the ball $K=K_R$, which is a unique solution of the equation (30). Obviously, this solution is also the unique solution of the system (18), (20) in the ball $K=K_R$.

From the structure of the space $B_{2,T}^3$ it follows that the functions $u(x,t)$, $u_x(x,t)$ and $u_{xx}(x,t)$ are continuous in the domain D_T.

By (24), it is easily seen from (29) that

$$\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq\sqrt{3}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}+\|p(t)\|_{C[0,T]}\left\|\|u_x(x,t)(1-2u(x,t))\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}.$$

The last relation implies that $u_t(x,t)$ is continuous in D_T.

It is not difficult to verify that the equation (2) and the conditions (3)-(5) and (7) are fulfilled in the usual sense. Thus, the solution of the problem (2)-(4), (7) is a pair of functions $\{u(x,t),p(t)\}$. According to the corollary of Lemma, this solution is unique in the ball $K=K_R$. The theorem is proved. ◀

From Theorems 1 and 2, we obtain unique solvability of the problem (2)-(5).

Theorem 3. *Let the conditions of Theorem 2 be fulfilled,* $0 \le b(t)\,(0 \le t \le T)$ *and the condition of coordination*

$$\varphi(1) = h(0) + \int_0^T b(t)h(t)dt$$

holds. Then the problem (2)-(5) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^3} \le R = A(T) + 2)$ *of the space* E_T^3.

References

1. Azizbayov E., Mehraliyev Y. Solvability of nonlocal inverse boundary-value problem for a second-order parabolic equation with integral conditions. *Electron. J. Differ. Equ.*, 2017, **2017** (125), pp. 1–14.
2. Fisher R.A. The wave of advance of advantageous genes. *Annals of Eugenics*, 1937, **7** (4), pp. 355–369.
3. Gamzaev Kh.M. A numerical method of solving the coefficient inverse problem for the nonlinear equation of diffusion-reaction. *Vestnik YuUrGU. Ser. Mat. Model. Progr.*, 2018, **11** (1), pp. 145–151 (in Russian).
4. Ivanchov M.I. *Inverse Problem for Equations of Parabolic Type.* VNTL Publishers, Lviv, Ukraine, 2003.
5. Kamynin V.L. The inverse problem of determining the lower-order coefficient in parabolic equations with integral observation. *Math. Notes*, 2013, **94** (2), pp. 205–213.
6. Kerimov N.B., Ismailov M.I. An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions. *J. Math. Anal. Appl.*, 2012, **396** (2), pp. 546–554.
7. Khudaverdiyev K.I., Veliyev A.A. *The Study of One-Dimensional Mixed Problem for One Class of Third Order Pseudo-Hyperbolic Equations with Nonlinear Operator Right-Hand Side.* Çaşıoğlu, Baku, 2010 (in Russian).
8. Kolmogorov A.N., Petrovsky I.G., Piskunov N.S. Research of the equation of the diffusion connected to increase of quantity of substance, and its application to one biological problem. *Moscow Univ. Math. Bulletin*, 1937, **1** (6), pp. 1–26 (in French).
9. Kozhanov A.I. Parabolic equations with an unknown time-dependent coefficient. *Comput. Math. Math. Phys.*, 2005, **45** (12), pp. 2085–2101.