Baku Mathematical Journal
2022, Vol. 1, No 2, PP. 179-194
https://doi.org/10.32010/j.bmj.2022.19

INVERSE STURM–LIOUVILLE PROBLEMS WITH POLYNOMIALS IN NONSEPARATED BOUNDARY CONDITIONS

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Received: 07.06.2022 / Revised: 18.07.2022 / Accepted: 25.07.2022

Abstract. *An nonself-adjoint Sturm–Liouville problem with two polynomials in nonseparated boundary conditions are considered. It is shown that this problem have an infinite countable spectrum. The corresponding inverse problems is solved. Criterions for unique reconstruction of the nonself-adjoint Sturm-Liouville problem by eigenvalues of this problem and the spectral data of an additional problem with separated boundary conditions are proved. Schemes for unique reconstruction of the Sturm-Liouville problems with polynomials in nonseparated boundary conditions and corresponding examples are given.*

Keywords: Sturm–Liouville problem, nonseparated boundary conditions, inverse problems, eigenvalues, spectral data

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

Let L denote the Sturm–Liouville problem

$$ly = -y'' + q(x)\,y = \lambda\,y = s^2\,y, \tag{1}$$

$$U_1(y) = -h\,y(0) + y'(0) + a(\lambda)\,y(\pi) = 0, \tag{2}$$

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$$U_2(y) = b(\lambda)\, y(0) + (-H_2 + H\,\lambda)\, y(\pi) + (\lambda - H_1)\, y'(\pi) = 0, \tag{3}$$

where $q(x) \in L_2(0,\pi)$ is a real-valued function; $h, H, H_1, H_2 \in \mathbb{R}$;

$$\rho := H\,H_1 - H_2 > 0; \tag{4}$$

the functions $a(\lambda)$ and $b(\lambda)$ are polynomials.

The inverse Sturm-Liouville problem for L in the case of separated boundary conditions without polinomials have been well studied (see [5], [8], [9]). The inverse Sturm-Liouville problem for L with spectral parameter in separated boundary conditions have been studied in [1], [3], [6], [7], [10]. The inverse Sturm-Liouville problem with unknown coefficients in nonseparated boundary conditions was studied by V.A. Sadovnichii, V.A. Yurko, V.A. Marchenko, O.A. Plaksina, M.G. Gasymov, I.M. Guseinov, I.M. Nabiev, and other authors (see [2], [4], [11]-[14], [16]-[18]). For nonself-adjoint SturmLiouville problem with polinomials in nonseparated boundary conditions no uniqueness theorems have been obtained.

The analysis of the inverse nonself-adjoint problem Sturm–Liouville with nonseparated boundary conditions was initiated in [13]. It was shown there that three spectra and two sets of weight numbers and residues of certain functions are sufficient for the unique reconstruction of a nonself-adjoint Sturm–Liouville problem with nonseparated boundary conditions. Moreover, these spectral data were used essentially [15]. Later, there were attempts to choose the problem to be reconstructed or auxiliary problems so as to use less spectral data for the reconstruction [2], [11]–[14], [16]–[18]. In particular, in [11] a nonself-adjoint problem was replaced by a self-adjoint one, and it was shown that, for its unique reconstruction, as spectral data it suffices to use three spectra, some sequence of signs, and some real number. In [2], an auxiliary problem was chosen so as to reduce the number of spectral data required for the reconstruction of a self-adjoint problem by only two spectra, some sequence of signs, and some real number were used as spectral data. In [16] a nonself-adjoint Sturm–Liouville problem was uniquely reconstructed by three spectrum. In the present paper, we consider a nonself-adjoint Sturm–Liouville problem with polinomials in nonseparated boundary conditions. We show that, for its unique reconstruction, one can use also less spectral data as compared with the reconstruction of a self-adjoint problem in [2], [11], [13]; more precisely, we need finite eigenvalues of the nonself-adjoint Sturm–Liouville problem, and, in addition, a spectrum and norming constants of an additional problem with separated boundary conditions.

2. Spectrum of L

We denote the matrix composed of the coefficients a_{lk} in the boundary conditions (2), (3) by A:

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{Vmatrix}$$

and its minors composed of the i-th and j-th columns by M_{ij}:

$$M_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix},$$

where

$$a_{11} = -h, \quad a_{12} = 1, \quad a_{13} = a(\lambda), \quad a_{14} = 0,$$

$$a_{21} = b(\lambda), \quad a_{22} = 0, \quad a_{23} = -H_2 + H\,\lambda, \quad a_{24} = \lambda - H_1.$$

Note that $\operatorname{rank} A = 2$, since $M_{24} = \lambda - H_1 \neq 0$.

Theorem 1. *If condition (4) holds, then problem L have an infinite countable spectrum.*

Proof. The eigenvalues of Problem L are the roots of the entire function [8, pp. 26-27], [9, pp. 14-15]

$$\Delta(\lambda) = M_{12} + M_{34} + M_{32}\, y_1(\pi, \lambda) + M_{42}\, y_1'(\pi, \lambda) +$$
$$+ M_{13}\, y_2(\pi, \lambda) + M_{14}\, y_2'(\pi, \lambda),$$

where $y_1(x, \lambda)$ $y_2(x, \lambda)$ are linearly independent solutions to differential equation (1) satisfying the conditions

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1. \tag{5}$$

Therefore,

$$\Delta(\lambda) = -b(\lambda) + a(\lambda)\,(\lambda - H_1) + (H_2 - H\,\lambda)\, y_1(\pi, \lambda) + (H_1 - \lambda)\; y_1'(\pi, \lambda) +$$
$$+ (h\,H_2 - h\,H\,\lambda - a(\lambda)\, b(\lambda))\, y_2(\pi, \lambda) + h\,(H_1 - \lambda)\, y_2'(\pi, \lambda).$$

For problem L we have the following alternatives [9, p. 14].

1. Every number λ is an eigenvalue for L;

2. Problem L has at most denumerably many eigenvalues (in particular cases, none at all), and these eigenvalues can have no finite limit-point.

Let us to prove the zero set of the all eigenvalues of problem L can not be $\mathbb{C}$, finite or empty.

The following asymptotic relations hold [9, pp. 52-54]:

$$\begin{array}{l} y_1(x, \lambda) = \cos sx + \frac{1}{s}\, u(x)\, \sin sx + \mathcal{O}\left(\frac{1}{s^2}\right), \\ y_2(x, \lambda) = \frac{1}{s}\, \sin sx - \frac{1}{s^2}\, u(x)\, \cos sx + \mathcal{O}\left(\frac{1}{s^3}\right), \\ y_1'(x, \lambda) = -s\, \sin sx + u(x)\, \cos sx + \mathcal{O}\left(\frac{1}{s}\right), \\ y_2'(x, \lambda) = \cos sx + \frac{1}{s}\, u(x)\, \sin sx + \mathcal{O}\left(\frac{1}{s^2}\right), \end{array} \tag{6}$$

where $u(x) = \frac{1}{2} \int\limits_0^x q(t)\, dt$, for sufficiently large $\lambda \in \mathbb{R}$ ([9, pp. 62–65]).

The relation $y_1(\pi, \lambda) = y_2'(\pi, \lambda)$ holds if and only if $q(x) = q(x - \pi)$ [17, Lemma 4].

If $q(x) = q(\pi - x)$, then

$$\Delta(\lambda) = -b(\lambda) + a(\lambda)\,(\lambda - H_1) + (H_2 - H\,\lambda + h\,(H_1 - \lambda))\, y_1(\pi, \lambda) +$$
$$+ (H_1 - \lambda)\; y_1'(\pi, \lambda) + (h\,H_2 - h\,H\,\lambda - a(\lambda)\, b(\lambda))\, y_2(\pi, \lambda). \tag{7}$$

It follows from (6) and (7) that the zero set of the function $\Delta(\lambda)$ is finite iff the following conditions hold:

$$-b(\lambda) + a(\lambda)\,(\lambda - H_1) \not\equiv C = \text{const}, \tag{8}$$

$$(H_2 - H\,\lambda + h\,(H_1 - \lambda))\,y_1(\pi,\,\lambda) + (H_1 - \lambda)\;y_1'(\pi,\,\lambda) + \\ +(h\,H_2 - h\,H\,\lambda - a(\lambda)\,b(\lambda))\,y_2(\pi,\,\lambda) \equiv 0. \tag{9}$$

It follows from (6) and (7) that the zero set of the function $\Delta(\lambda)$ is $\mathbb{C}$ or empty if the following conditions hold: (9) and

$$-b(\lambda) + a(\lambda)\,(\lambda - H_1) \equiv C = \text{const}. \tag{10}$$

Since condition (4) holds we see that $(H_2 - H\,\lambda + h\,(H_1 - \lambda)) \neq 0$. Indeed, if $(H_2 - H\,\lambda + h\,(H_1 - \lambda)) = 0$, then $h + H = 0$ and $H_2 + h\,H_1 = 0$. From this we obtain $h = -H$ and $H_2 - H\,H_1 = 0$. This contradicts condition (4). The contradiction proves inequalities $(H_2 - H\,\lambda + h\,(H_1 - \lambda)) \neq 0$ and $(H_2 - H\,\lambda + h\,(H_1 - \lambda))\,y_1(\pi,\,\lambda) \not\equiv 0$.

Since $(H_2 - H\,\lambda + h\,(H_1 - \lambda))\,y_1(\pi,\,\lambda) \not\equiv 0$ and asymptotic relations (6) hold, we see that identity (9) is not true. So for case $q(x) = q(\pi - x)$ the zero set of the function $\Delta(\lambda)$ can not be $\mathbb{C}$, finite or empty. This completes the proof of Theorem 1. ◄

Let be $q(x) \neq q(\pi - x)$. It follows from (6) and (7) that the zero set of the function $\Delta(\lambda)$ is finite iff the following conditions hold: (8) and

$$(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + (H_1 - \lambda)\;y_1'(\pi,\,\lambda) + (h\,H_2 - h\,H\,\lambda - a(\lambda)\,b(\lambda))\,y_2(\pi,\,\lambda) + \\ +h\,(H_1 - \lambda)\,y_2'(\pi,\,\lambda) \equiv 0. \tag{11}$$

If condition (4) holds, then $(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1 - \lambda)\,y_2'(\pi,\,\lambda) \not\equiv 0$. Assume the converse. Then we have $(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1 - \lambda)\,y_2'(\pi,\,\lambda) \equiv 0$. From asymptotic relations (6) it follows $h + H = 0$ (the elder coefficient must be zero). If $h + H = 0$, then the elder coefficient is $H_2 + h\,H_1$. It must be zero too. From this we obtain $h = -H$ and $H_2 - H\,H_1 = 0$. This contradicts condition (4). As above the contradiction proves the zero set of the function $\Delta(\lambda)$ can not be $\mathbb{C}$, finite or empty.

From the proof we see that Theorem 1 will be true if we replace condition (4) by the following condition:

$$H\,H_1 - H_2 \neq 0.$$

3. Inverse Problem for L

Together with problem L, we consider the following problem with separated boundary conditions.

Problem L_1. The Equation (1) with separated boundary conditions

$$U_1(y) = -h\,y(0) + y'(0) = 0,$$

$$U_2(y) = (-H_2 + H\,\lambda)\,y(\pi) + (\lambda - H_1)\,y'(\pi) = 0.$$

Let the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1) under the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h,$$

$$\psi(\pi,\lambda) = H_1 - \lambda, \quad \psi'(\pi,\lambda) = \lambda\,H - H_2.$$

By definition, put

$$\chi(\lambda) = \varphi(x,\lambda)\,\psi'(x,\lambda) - \varphi'(x,\lambda)\,\psi(x,\lambda),$$

which is independent of $x \in [0,\pi]$. The function $\chi(\lambda)$ is entire and has zeros at the eigenvalues μ_n of Problem L_1. The set of eigenvalues μ_n of Problem L_1 is countable, consists of real numbers and for each eigenvalue $\mu_n$there exists such a number $k_n \neq 0$ that $\psi(x,\mu_n) = k_n\,\varphi(x,\mu_n)$.

Numbers γ_n are *norming constants* if [3]:

$$\gamma_n = \int_0^\pi \varphi^2(x,\mu_n)\,dx + \frac{\left(\varphi'(\pi,\mu_n) + H\,\varphi(\pi,\mu_n)\right)^2}{\rho}.$$

The numbers $\{\mu_n,\,\gamma_n\}$ are called *spectral data* of problem L_1.

For problem L, we pose the inverse problem.

Inverse problem. Suppose that the potential function $q(x)$ and the coefficients in the boundary conditions of problem L are unknown, while the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1 and eigenvalues λ_m of problem L are known. It is required to find $q(x)$ and boundary conditions of problem L from the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1 and eigenvalues λ_m of problem L.

4. A Criterion for Unique Reconstruction of Problem L by the Spectral Data of Problem L_1 and All Eigenvalues of Problem L

Theorem 2. *Suppose condition (4) holds. Then problem L can be uniquely reconstructed from the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1 and all eigenvalues λ_m of problem L if and only if the polynomials $a(\lambda)$ and $b(\lambda)$ have the form*

$$a(\lambda) = a_0\,\lambda^{m_0} + a_1\,\lambda^{m_1} + \cdots + a_{n-1}\,\lambda^{m_{n-1}}, \quad b(\lambda) = b_0\,\lambda^{p_0} + b_1\,\lambda^{p_1} + \cdots + b_{n-1}\,\lambda^{p_{n-1}},$$

where

$$m_i + 1 \neq p_j \quad \textit{for all} \;\; i,j = 1,\,2,\,\ldots,\,n-1. \tag{12}$$

Proof. It follows from [3] that if $H\,H_1 - H_2 > 0$, then the function $q(x)$ and the numbers h, H, H_1, and H_2 is uniquely determined by the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1.

It remains to check that the polynomials $a(\lambda)$ and $b(\lambda)$ are uniquely reconstructed from the eigenvalues λ_m of problem L if condition (12) holds, and the polynomials $a(\lambda)$ and $b(\lambda)$ can not be uniquely reconstructed from the eigenvalues λ_m of problem L if condition (12) does not hold.

Suppose there exist polynomials $\widetilde{a}(\lambda)$ and $\widetilde{b}(\lambda)$ for which the spectrum of problem L have the same eigenvalues λ_m.

Since spectrum of Problem L is countable we see that according to Hadamards theorem, the function $\Delta(\lambda)$ (which is entire of order 1/2) can be reconstructed from its zeros up to a factor $C \neq 0$. Therefore, the functions $\Delta(\lambda)$ for the polynomials $a(\lambda)$ and $b(\lambda)$, and $\widetilde{\Delta}(\lambda)$ polynomials $\widetilde{a}(\lambda)$ and $\widetilde{b}(\lambda)$ are related by the identity

$$\widetilde{\Delta}(\lambda) \equiv C\,\Delta(\lambda), \tag{13}$$

where C is a nonzero constant.

If $q(x) = q(\pi - x)$, then from (13) and (7) we obtain the following identity

$$\begin{aligned}\widetilde{\Delta}(\lambda) - C\,\Delta(\lambda) \equiv -\Big(\widetilde{b}(\lambda) - C\,b(\lambda)\Big) + \Big(\widetilde{a}(\lambda) - C\,a(\lambda)\Big)\,(\lambda - H_1) + \\ +(1-C)\,(H_2 - H\,\lambda + h\,(H_1-\lambda))\,y_1(\pi,\,\lambda) + \\ +(1-C)\,(H_1-\lambda)\;y_1'(\pi,\,\lambda) + (1-C)\,(h\,H_2 - h\,H\,\lambda)\,y_2(\pi,\,\lambda) - \\ -\Big(\widetilde{a}(\lambda)\,\widetilde{b}(\lambda) - C\,a(\lambda)\,b(\lambda)\Big)\,y_2(\pi,\,\lambda).\end{aligned} \tag{14}$$

If condition (4) holds, then $(H_2 - H\,\lambda + h\,(H_1-\lambda))\,y_1(\pi,\,\lambda) \not\equiv 0$. (See the proof of theorem 1.) Since $(H_2 - H\,\lambda + h\,(H_1-\lambda))\,y_1(\pi,\,\lambda) \not\equiv 0$ and asymptotic relations (6) hold, we see that the functions 1, $(H_2 - H\,\lambda + h\,(H_1-\lambda))\,y_1(\pi,\,\lambda)$, and $y_2(\pi,\,\lambda)$ are polynomially independent. (We say that the functions $f_1(\lambda)$, $f_2(\lambda)$, $\ldots$, $f_n(\lambda)$ are polynomially independent if their combination

$$P_1(\lambda)\,f_1(\lambda) + P_2(\lambda)\,f_2(\lambda) + \cdots + P_n(\lambda)\,f_n(\lambda)$$

with the arbitrary polynomials $P_1(\lambda)$, $P_2(\lambda)$, $\ldots$, $P_n(\lambda)$ is identically equal to zero only in the case when $P_k(\lambda) \equiv 0$ $(k = 1, 2, \ldots, n)$.) From this, (12) and (14) we have

$$1 - C = 0, \quad \widetilde{a}(\lambda) = a(\lambda), \quad \widetilde{b}(\lambda) = b(\lambda), \tag{15}$$

$$\begin{aligned}\Delta(\lambda) = -b(\lambda) + a(\lambda)\,(\lambda - H_1) + (H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + (H_1-\lambda)\;y_1'(\pi,\,\lambda) + \\ +(h\,H_2 - h\,H\,\lambda - a(\lambda)\,b(\lambda))\,y_2(\pi,\,\lambda) + h\,(H_1-\lambda)\,y_2'(\pi,\,\lambda).\end{aligned} \tag{16}$$

If $q(x) \neq q(\pi - x)$, from (11) and (13) we get

$$\begin{aligned}\widetilde{\Delta}(\lambda) - C\,\Delta(\lambda) \equiv -\Big(\widetilde{b}(\lambda) - C\,b(\lambda)\Big) + \Big(\widetilde{a}(\lambda) - C\,a(\lambda)\Big)\,(\lambda - H_1) + \\ +(1-C)\,\Big((H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1-\lambda)\,y_2'(\pi,\,\lambda)\Big) + \\ +(1-C)\,(H_1-\lambda)\;y_1'(\pi,\,\lambda) + (1-C)\,(h\,H_2 - h\,H\,\lambda)\,y_2(\pi,\,\lambda) - \\ -\Big(\widetilde{a}(\lambda)\,\widetilde{b}(\lambda) - C\,a(\lambda)\,b(\lambda)\Big)\,y_2(\pi,\,\lambda).\end{aligned} \tag{17}$$

If condition (4) holds, then $(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1-\lambda)\,y_2'(\pi,\lambda) \not\equiv 0$. (See the proof of theorem 1.) Since $(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1-\lambda)\,y_2'(\pi,\,\lambda) \not\equiv 0$ and asymptotic relations (6) hold, we see that the functions 1, $(H_2 - H\,\lambda)\,y_1(\pi,\,\lambda) + h\,(H_1-\lambda)\,y_2'(\pi,\,\lambda)$, and $y_2(\pi,\,\lambda)$ are polynomially independent. From this, (12) and (17) we get (15). Thus, Problem L can be uniquely reconstructed from the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1 and all eigenvalues λ_m of problem L. If condition (12) holds the theorem is proved. ◄

Suppose condition (12) does not hold. It follows from linear independence of core-sponding functions that $C = 1$ (see above). If condition (12) does not hold there exist $i = i_0$ and $j = j_0$ such that $m_{i_0} + 1 = p_{j_0}$. Then for all a_{i_0} and b_{j_0} such that $a_{i_0} - b_{j_0} = 0$ identity (14) or (17) holds (as $C = 1$). This proves that in this case the problem of reconstructing the polinomials $a(\lambda)$ and $b(\lambda)$ has nonunique solution.

Remark 1. If problem L is a spectral problem with separated boundary conditions ($a(\lambda) \equiv b(\lambda) \equiv 0$), then it coincides with problem L_1. By taking the values μ_k for λ_k, we find that N.J. Guliyev's Theorem 4.1 [3] is a special case of Theorem 2.

Example 1. If $H_1 = 0$, $a(\lambda) \equiv a$, $b(\lambda) \equiv b\,\lambda$, then from (14) we see that Problem L have the same eigenvalues for all a and b, such that $a = b$.

In next two sections we consider special cases of Condition (12):
(i) the polynomial $a(\lambda)$ is unknown, the polynomial $b(\lambda)$ is known;
(ii) the polynomial $a(\lambda)$ is known, the polynomial $b(\lambda)$ is unknown;
(iii) $b(\lambda) = b_0 + b_1\,\lambda + a_2\,\lambda^2 + \cdots + b_{n-1}\,\lambda^{n-1}$, $a(\lambda) = a_n\,\lambda^n + a_{n+1}\,\lambda^{n+1} + \cdots + a_{2n-1}\,\lambda^{2n-1}$.
For unique reconstruction of Problem L in these cases we use and the spectral data of problem L_1 and only a finite set of eigenvalues of Problem L.

5. Criterions for Uniquely Reconstruction Problem L with One Unknown Polinomial by the Spectral Data of Problem L_1 and Finite Set of Eigenvalues of Problem L

In this section we consider the following special cases of Condition (12):
(i) the polynomial $a(\lambda)$ is unknown, the polynomial $b(\lambda)$ is known;
(ii) the polynomial $a(\lambda)$ is known, the polynomial $b(\lambda)$ is unknown.
For unique reconstruction of Problem L in these cases we use and the spectral data of problem L_1 and only a finite set of eigenvalues of Problem L.

Theorem 3. *Suppose condition (4) holds, the polynomial $b(\lambda)$ and the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L are known; and the potential function $q(x)$, the numbers h, H, H_1, H_2, and polynomial*

$$a(\lambda) = a_0 + a_1\,\lambda + a_2\,\lambda^2 + \cdots + a_{n-1}\,\lambda^{n-1}$$

are unknown. Then problem L is uniquely reconstructed from the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1 and n non-zero mutually different eigenvalues λ_k ($k = 1, 2, \ldots, n$) if and only if the following condition

$$b(\lambda_k)\,y_2(\pi, \lambda_k) - \lambda_k + H_1 \neq 0, \qquad k = 1, 2, \ldots, n, \tag{18}$$

holds (The procedure for constructing the function $y_2(\pi, \lambda)$ from the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1 is clear from proof the theorem.).

Proof. It follows from [3] that if $H\,H_1 - H_2 > 0$, then the function $q(x)$ and the numbers h, H, H_1, and H_2 is uniquely determined by the spectral data $\{\mu_n,\,\gamma_n\}$ of problem L_1. The procedure for recovering the function $q(x)$ and the numbers h, H, H_1, and H_2 is given in [3]. Since the function $q(x)$ is known, we can find the solution $y_2(\pi,\,\lambda)$ to differential equation (1) satisfying the conditions (5) and consider condition (18).

It remains to check that the polynomials $a(\lambda)$ is uniquely reconstructed from the eigenvalues λ_k ($k = 1,\,2,\,\ldots,\,n$) of problem L if condition (18) holds, and the polynomials $a(\lambda)$ is not uniquely reconstructed from the eigenvalues λ_k ($k = 1,\,2,\,\ldots,\,n$) of problem L if condition (18) does not hold.

Suppose condition (18) holds. Substituting the known eigenvalues of problem L into (16), we obtain a system of algebraic equations for unknown coefficients $a_0,\,a_1,\,a_2,\,\ldots,\,a_{n-1}$:

$$\Delta(\lambda_k) = -b(\lambda_k) + a(\lambda_k)\,(\lambda_k - H_1) + (H_2 - H\,\lambda_k)\,y_1(\pi,\,\lambda_k) + (H_1 - \lambda_k)\;y_1'(\pi,\,\lambda_k) +$$

$$+ (h\,H_2 - h\,H\,\lambda_k - a(\lambda_k)\,b(\lambda_k))\,y_2(\pi,\,\lambda_k) + h\,(H_1 - \lambda_k)\,y_2'(\pi,\,\lambda_k) = 0,$$

or

$$a_0 + a_1\,\lambda_k + a_2\,\lambda_k^2 + \cdots + a_{n-1}\,\lambda_k^{n-1} =$$

$$= (b(\lambda_k)\,y_2(\pi,\,\lambda_k) - \lambda_k + H_1)^{-1}\Big(- b(\lambda_k) + (H_2 - H\,\lambda_k)\,y_1(\pi,\,\lambda_k) +$$

$$+ (H_1 - \lambda_k)\;y_1'(\pi,\,\lambda_k) + (h\,H_2 - h\,H\,\lambda_k)\,y_2(\pi,\,\lambda_k) + h\,(H_1 - \lambda_k)\,y_2'(\pi,\,\lambda_k)\Big), \qquad (19)$$

where $k = 1,\,2,\,\ldots,\,n$.

The determinant of system (19) w.r.t. unknowns $a_0,\,a_1,\,a_2,\,\ldots,\,a_{n-1}$, is the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ & & \ldots & \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{vmatrix} = (\lambda_n - \lambda_{n-1})\ldots(\lambda_n - \lambda_1)\ldots(\lambda_2 - \lambda_1) \neq 0.$$

Hence, system of equations (10) has the unique solution determined, for example, by Cramers formulae:

$$a_0 = \frac{\Delta_1}{\Delta},\ldots,a_{n-1} = \frac{\Delta_n}{\Delta},$$

where the determinants Δ_k ($k = 1,\,\ldots,\,n$) are obtained from determinant Δ by replacing ith column by the column of the right hand sides in system of equations (19). For case condition (18) holds the theorem is proved. ◀

Suppose condition (18) does not hold. Then among n non-zero mutually different eigenvalues λ_k ($k = 1,\,2,\,\ldots,\,n$) there exists the eigenvalue $\lambda_k = \nu$ such that

$$b(\nu)\,y_2(\pi,\,\nu) - \nu + H_1 = 0.$$

Without loss of generality, we can assume that $\nu = \lambda_1$. Substituting the known eigenvalues of problem L into (16), we obtain a system of algebraic equations for unknown coefficients $a_0, a_1, a_2, \ldots, a_{n-1}$:

$$\begin{gathered}(b(\lambda_k)\, y_2(\pi,\, \lambda_k) - \lambda_k + H_1)\,(a_0 + a_1\,\lambda_k + a_2\,\lambda_k^2 + \cdots + a_{n-1}\,\lambda_k^{n-1}) = \\ = \Big(- b(\lambda_k) + (H_2 - H\,\lambda_k)\, y_1(\pi,\, \lambda_k) + \\ +(H_1 - \lambda_k)\; y_1'(\pi,\, \lambda_k) + (h\, H_2 - h\, H\,\lambda_k)\, y_2(\pi,\, \lambda_k) + h\,(H_1 - \lambda_k)\, y_2'(\pi,\, \lambda_k)\Big),\end{gathered} \tag{20}$$

where $k = 1, 2, \ldots, n$.

Since $(b(\lambda_1)\, y_2(\pi, \lambda_1) - \lambda_1 + H_1) = 0$, we see that the first equation of this system does not consist $(a_0 + a_1\,\lambda_1 + a_2\,\lambda_1^2 + \cdots + a_{n-1}\,\lambda_1^{n-1})$. System of linear algebraic equations (20) has n unknowns and $n-1$ equations. So the system of equations (20) has nonunique solution. This completes the proof of Theorem 3.

Theorem 4. *Suppose condition (4) holds, the polynomial $a(\lambda)$ and the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L are known; and the potential function $q(x)$, the numbers h, H, H_1, H_2, and polynomial*

$$b(\lambda) = b_0 + b_1\,\lambda + b_2\,\lambda^2 + \cdots + b_{n-1}\,\lambda^{n-1}$$

are unknown. Then problem L can be uniquely reconstructed from the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1 and n non-zero mutually different eigenvalues λ_k if and only if the following condition

$$1 + a(\lambda_k)\, y_2(\pi,\, \lambda_k) \neq 0, \qquad k = 1, 2, \ldots, n, \tag{21}$$

holds.

Proof. It follows from [3] that if $H\,H_1 - H_2 > 0$, then the function $q(x)$ and the numbers h, H, H_1, and H_2 is uniquely determined by the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1. The procedure for recovering the function $q(x)$ and the numbers h, H, H_1, and H_2 is given in [3]. Since the function $q(x)$ is known, we can find the solutions $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ to differential equation (1) satisfying the conditions (5) and consider condition (21).

It remains to check that the polynomials $b(\lambda)$ is uniquely reconstructed from the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L if condition (21) holds, and the polynomials $b(\lambda)$ is not uniquely reconstructed from the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L if condition (21) does not hold.

Suppose condition (21) holds. Substituting the known eigenvalues of problem L into (16), we obtain a system of algebraic equations for unknown coefficients $b_0, b_1, b_2, \ldots, b_{n-1}$:

$$\begin{gathered}b_0 + b_1\,\lambda_k + b_2\,\lambda_k^2 + \cdots + b_{n-1}\,\lambda_k^{n-1} = \\ = (1 + a(\lambda_k)\, y_2(\pi,\, \lambda_k))^{-1}\,\Big(a(\lambda_k)\,(\lambda_k - H_1) + (H_2 - H\,\lambda_k)\, y_1(\pi,\, \lambda_k) + \\ +(H_1 - \lambda_k)\; y_1'(\pi,\, \lambda_k) + (h\, H_2 - h\, H\,\lambda_k)\, y_2(\pi,\, \lambda_k) + h\,(H_1 - \lambda_k)\, y_2'(\pi,\, \lambda_k)\Big) = 0,\end{gathered} \tag{22}$$

where $k = 1, 2, \dots, n$.

The determinant of system (22) w.r.t. unknowns $b_0, b_1, b_2, \dots, b_{n-1}$, is the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ & & \dots & \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{vmatrix} = (\lambda_n - \lambda_{n-1}) \dots (\lambda_n - \lambda_1) \dots (\lambda_2 - \lambda_1) \neq 0.$$

Hence, system of equations (10) has the unique solution determined, for example, by Cramers formulae:

$$b_0 = \frac{\Delta_1}{\Delta}, \dots, b_{n-1} = \frac{\Delta_n}{\Delta},$$

where the determinants Δ_k ($k = 1, \dots, n$) are obtained from determinant Δ by replacing ith column by the column of the right hand sides in system of equations (22). For case condition (21) holds the theorem is proved.

Suppose condition (21) does not hold. Then among n non-zero mutually different eigenvalues λ_k ($k = 1, 2, \dots, n$) there exists the eigenvalue $\lambda_k = \nu$ such that

$$1 + a(\nu)\, y_2(\pi, \nu) = 0.$$

Without loss of generality, we can assume that $\nu = \lambda_1$. Substituting the known eigenvalues of problem L into (16), we obtain a system of algebraic equations for unknown coefficients $b_0, b_1, b_2, \dots, b_{n-1}$:

$$(1 + a(\lambda_k)\, y_2(\pi, \lambda_k))\, (b_0 + b_1 \lambda_k + b_2 \lambda_k^2 + \dots + b_{n-1} \lambda_k^{n-1}) =$$
$$= a(\lambda_k)\,(\lambda_k - H_1) + (H_2 - H\lambda_k)\, y_1(\pi, \lambda_k) + (H_1 - \lambda_k)\; y_1'(\pi, \lambda_k) +$$
$$+ (h H_2 - h H \lambda_k)\, y_2(\pi, \lambda_k) + h\,(H_1 - \lambda_k)\, y_2'(\pi, \lambda_k), \qquad (23)$$

where $k = 1, 2, \dots, n$.

Since $1 + a(\lambda_1)\, y_2(\pi, \lambda_1) = 0$, we see that the first equation of this system does not consist $(b_0 + b_1 \lambda_k + b_2 \lambda_k^2 + \dots + b_{n-1} \lambda_k^{n-1})$. System of linear algebraic equations (20) has n unknowns and $n-1$ equations. So the system of equations (23) has nonunique solution. This completes the proof of Theorem 4. ◀

6. A Unique Reconstruction of Problem L by a Finite Set of Its Eigenvalues and the Spectral Data of Problem L_1

In this section we consider the following special case of Condition (12):
(iii) $b(\lambda) = b_0 + b_1 \lambda + a_2 \lambda^2 + \dots + b_{n-1} \lambda^{n-1}$, $a(\lambda) = a_n \lambda^n + a_{n+1} \lambda^{n+1} + \dots + a_{2n-1} \lambda^{2n-1}$.

For unique reconstruction of Problem L in these cases we use and the spectral data of problem L_1 and only a finite set of eigenvalues of Problem L.

Theorem 5. *Suppose condition (4) holds, the polynomial $b(\lambda)$ and the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L are known; and the potential function $q(x)$, the numbers h, H, H_1, H_2, and polynomials*

$$b(\lambda) = b_0 + b_1\,\lambda + a_2\,\lambda^2 + \cdots + b_{n-1}\,\lambda^{n-1}, \quad a(\lambda) = a_n\,\lambda^n + a_{n+1}\,\lambda^{n1} + \cdots + a_{2n-2}\,\lambda^{2n-1}$$

are unknown. Then problem L can be uniquely reconstructed from the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1 and n non-zero mutually different eigenvalues λ_k ($k = 1, 2, \ldots, n$) such that the determinant

$$|1, \ldots, \lambda_k^{n-1}, \lambda_k^n(\lambda_k - H_1), \ldots, \lambda_k^{2n-2}(\lambda_k - H_1), y_2(\pi, \lambda_k), \ldots, \lambda_k^{2n-2}\, y_2(\pi, \lambda_k)|_{k=1,\ldots,3n-1}. \tag{24}$$

is not equal to zero.

Proof. It follows from [3] that if $H\,H_1 - H_2 > 0$, then the function $q(x)$ and the numbers h, H, H_1, and H_2 is uniquely determined by the spectral data $\{\mu_n, \gamma_n\}$ of problem L_1.

It remains to check that the polynomials $a(\lambda)$ is uniquely reconstructed from the eigenvalues λ_k ($k = 1, 2, \ldots, n$) of problem L.

Denote by

$$x_i, \qquad i = 1, 2, \ldots, n,$$

the unknown coefficients b_i $(i = 0, 1, \ldots, n-1)$ of the polynomial $a(\lambda)$; denote by

$$x_i, \qquad i = n+1, n+2, \ldots, 2n,$$

the unknown coefficients $-a_i$ $(i = 0, 1, \ldots, n-1)$ of the polynomial $b(\lambda)$. Denote by

$$x_i, \qquad i = 2n+1, 2n+2, \ldots, 3n-1,$$

the different coefficients of the polynomial $a(\lambda)\,b(\lambda)$.

Substituting the $3n-1$ known eigenvalues of problem L into (16), we obtain a system of algebraic equations for unknown coefficients x_i $(i = 2n+1, 2n+2, \ldots, 3n-1)$:

$$x_1 + x_2\,\lambda_k + x_3\,\lambda_k^2 + \cdots + x_n\,\lambda_k^{n-1} +$$

$$+ (x_{n+1}\,\lambda_k^n + x_{n+2}\,\lambda_k^{n+1} + x_{n+3}\,\lambda_k^{n+2} + \cdots + x_{2n}\,\lambda_k^{2n-1})\,(\lambda_k - H_1) +$$

$$+ (x_{2n+1}\,\lambda_k^n + x_{2n+2}\;\lambda_k^{n+1} + x_{2n+3}\,\lambda_k^{n+2} + \cdots + x_{3n-1}\,\lambda_k^{3n-2})\,y_2(\pi,\,\lambda_k) =$$

$$= (H_2 - H\,\lambda_k)\,y_1(\pi,\,\lambda) + (H_1 - \lambda_k)\;y_1'(\pi,\,\lambda_k) +$$

$$+ (h\,H_2 - h\,H\,\lambda_k)\,y_2(\pi,\,\lambda_k) + h\,(H_1 - \lambda_k)\,y_2'(\pi,\,\lambda_k), \qquad k = 1, 2, \ldots 3n-1. \tag{25}$$

The determinant of system (25) w.r.t. unknowns x_i, $i = 1, ..., 3n-1$, is (24). If this determinant is not equal to zero the system of equations (25) has a unique solution. This completes the proof of Theorem 5. ◀

7. Examples

We consider using theorems 3, 4 and 5.

Procedure of identification of Problem L**.** On the basis of the proofs of Theorem 3, 4 and 5 one can construct an algorithm for the unique identification of problem L:

1. On the basis of spectral data $\{\mu_n, \gamma_n\}$ of problem L_1, we find an function $q(x)$ and numbers h, H, H_1, and H_2; i.e., we construct problems L_1. It is found with the use of well-known method of identification of a SturmLiouville problem with separated boundary conditions [3].

2. For the found function $q(x)$, we find linearly independent solutions $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ to differential equation (1) satisfying the conditions (5).

3. For the found polinomials $a(\lambda)$ and (or) $b(\lambda)$, we use methods from the proofs of theorems 3, 4 or 5. We thereby completely reconstruct problem L.

In all three examples, we assume that the spectral data of eigenvalue problem L_1 are the following:

$$\mu_0 = 0, \quad \mu_1 = \tfrac{1}{4}, \quad \mu_k = (k-1)^2, \qquad k \geq 2,$$
$$\gamma_0 = \pi, \quad \gamma_k = \tfrac{\pi}{2}, \qquad k \geq 1.$$

In this case, we have

$$q(x) = \frac{2(\pi + x)\sin x + 4(1+\cos x)}{(\pi + x + \sin x)^2}, \quad h = -\frac{2}{\pi}, \quad H = 0, \quad H_1 = \frac{1}{4}, \quad H_2 = -\frac{1}{8\pi}.$$

It follows from [3]. Below, for simplicity, we assume that these values have already been found at the step for the identification of problems L_1. In addition, we assume that linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of Equation (1) with condition (5) have been found. Then, in this case, we have (16).

Example 2. Suppose $a(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3$, $b(\lambda) \equiv 0$ and the eigenvalues of Problem L are the following:

$$\lambda_1 = -17.768; \quad \lambda_2 = 1.9118 - 1.0160i; \quad \lambda_3 = 1.9118 + 1.0160i; \quad \lambda_4 = 9.2515 - 29.228i.$$

Then the system of equations (19) has the following form

$$-18.018a_0 + 320.15a_1 - 5688.4a_2 + 1.0107 \cdot 10^5 a_3 = 3.878510^5,$$

$$(1.6618 - 1.0160\,i)\,a_0 + (2.1448 - 3.6308\,i)\,a_1 + (0.41148 - 9.1204\,i)\,a_2 -$$
$$-(8.4796 + 17.854\,i)\,a_3 = -26.733 - 107.06\,i,$$

$$(1.6618 + 1.0160\,i)\,a_0 + (2.1448 + 3.6308\,i)\,a_1 + (0.41148 + 9.1204\,i)\,a_2 +$$
$$+(-8.4796 + 17.854\,i)\,a_3 = -26.733 + 107.06\,i,$$

$$(9.0015 - 29.228\,i)\,a_0 + (-771.00 - 533.50\,i)\,a_1 + (-22726 + 17599\,i)\,a_2 +$$
$$+(3.0414 \cdot 10^5 + 8.2706 \cdot 10^5\,i)\,a_3 = 1.1468 \cdot 10^6 + 3.3599 \cdot 10^6\,i. \qquad (26)$$

System of equations (26) has the unique solution $a_0 = 1.0000$, $a_1 = 2.0000$, $a_2 = 3.0000$, $a_3 = 4.0000$. Consequently, $a(\lambda) = 1+2\,\lambda+3\,\lambda^2+4\,\lambda^3$ and problem L is following:

$$ly = -y'' + \frac{2\,(\pi + x)\,\sin x + 4\,(1 + \cos x)}{(\pi + x + \sin x)^2}\, y = \lambda\, y = s^2\, y,$$

$$\frac{2}{\pi}\, y(0) + y'(0) + (1 + 2\,\lambda + 3\,\lambda^2 + 4\,\lambda^3)\, y(\pi) = 0, \quad \frac{1}{8\,\pi}\, y(\pi) + \left(\lambda - \frac{1}{4}\right) y'(\pi) = 0.$$

Example 3. Suppose $a(\lambda) \equiv 0$, $b(\lambda) = b_0 + b_1\,\lambda + b_2\,\lambda^2 + b_3\,\lambda^3$ and the eigenvalues of problem L are the following:

$$\lambda_1 = -17.978; \quad \lambda_2 = 0.25345; \quad \lambda_3 = 1.9411; \quad \lambda_4 = 5.0550 + 3.7554\, i.$$

Then system of equations (22) has the following form

$$\begin{gathered} -b_0 + 17.978\, b_1 - 323.22\, b_2 + 5810.9\, b_3 = 11333, \\ -b_0 - 0.25345\, b_1 - 0.064237\, b_2 - 0.016281\, b_3 = -1.60370, \\ -b_0 - 1.9411\, b_1 - 3.7680\, b_2 - 7.3140\, b_3 = -23.278, \\ -b_0 - (5.0549 + 3.7554\, i)\, b_1 - (11.450 + 37.966\, i)\, b_2 + \\ +(84.698 - 234.92\, i)\, b_3 = 146.83 - 515.31\, i. \end{gathered} \tag{27}$$

System of equations (27) has the unique solution $b_0 = 1.0000$, $b_1 = 2.0000$, $b_2 = 1.0000$, $b_3 = 2.0000$. Consequently, $b(\lambda) = 1 + 2\,\lambda + \lambda^2 + 2\,\lambda^3$ and problem L is following:

$$ly = -y'' + \frac{2\,(\pi + x)\,\sin x + 4\,(1 + \cos x)}{(\pi + x + \sin x)^2}\, y = \lambda\, y = s^2\, y,$$

$$\frac{2}{\pi}\, y(0) + y'(0) = 0, \quad (1 + 2\,\lambda + \lambda^2 + 2\,\lambda^3)\, y(0) + \frac{1}{8\,\pi}\, y(\pi) + \left(\lambda - \frac{1}{4}\right) y'(\pi) = 0.$$

Example 4. Suppose $a(\lambda) = a_2\,\lambda^2 + a_3\,\lambda^3$, $b(\lambda) = b_0 + b_1\,\lambda$ and the eigenvalues of problem L are the following:

$$\lambda_1 = 0.29110; \quad \lambda_2 = 0.66347; \quad \lambda_3 = 2.7751; \quad \lambda_4 = 7.6014 - 2.2275\, i;$$

$$\lambda_5 = 7.6014 + 2.2275\, i; \quad \lambda_6 = 10.666 - 50.994\, i; \quad \lambda_7 = 10.666 - 50.994\, i.$$

Then the system of equations (25) has the following form

$$\begin{gathered} -x_1 - 0.29109\, x_2 + 0.0034822\, x_3 + 0.0010137\, x_4 - \\ -1.3315\, x_5 - 0.38759\, x_6 - 0.11283\, x_7 = -10.341, \\ -x_1 - 0.66348\, x_2 + 0.18201\, x_3 + 0.12076\, x_4 - \\ -4.8617\, x_5 - 3.2256\, x_6 - 2.1401\, x_7 = -65.260, \\ -x_1 - 2.7751\, x_2 + 19.447\, x_3 + 53.968\, x_4 - 2.4411\, x_5 - 6.7743\, x_6 - 18.800\, x_7 = 42.199, \end{gathered}$$

$$-x_1 + (-7.6013 + 2.2275\,i)\,x_2 + (312.85 - 366.61\,i)\,x_3 + (1561.5 - 3483.7\,i)\,x_4 +$$

$$+(-23.986 + 16.072\,i)\,x_5 + (-146.52 + 175.60\,i)\,x_6 +$$

$$+(-722.61 + 1661.2\,i)\,x_7 = -149.57 + 63.846\,i,$$

$$-x_1 + (-7.6013 - 2.2275\,i)\,x_2 + (312.85 + 366.61\,i)\,x_3 + (1561.5 + 3483.7\,i)\,x_4 +$$

$$+(-23.986 - 16.072\,i)\,x_5 + (-146.52 - 175.60\,i)\,x_6 +$$

$$+(-722.61 - 1661.2\,i)\,x_7 = -149.57 - 63.846\,i,$$

$$-x_1 + (-10.666 + 50.994i)x_2 + (-81374 + 11547 \cdot 10i)x_3 + (5.0204 + 5.3812i) \cdot 10^6\,x_4 +$$

$$+(-30020 + 48413\,i)\,x_5 + (2.1485 + 2.0472\,i) \cdot 10^6\,x_6 +$$

$$+(12.731 - 8.7729\,i) \cdot 10^7\,x_7 = (1.0597 - 0.65934\,i) \cdot 10^9,$$

$$-x_1 + (-10.666 - 50.994\,i)\,x_2 + (-81374 - 11547 \cdot 10\,i)\,x_3 +$$

$$+(5.0204 - 5.3812\,i) \cdot 10^6\,x_4 +$$

$$+(-30020 - 48413\,i)\,x_5 + (2.1485 - 2.0472\,i) \cdot 10^6\,x_6 +$$

$$+(12.731 + 8.7729\,i) \cdot 10^7\,x_7 = (1.0597 + 0.65934\,i) \cdot 10^9.$$

This system of equations has the unique solution $x_1 = 1.0000$, $x_2 = 2.0000$, $x_3 = 3.0000$, $x_4 = 4.0000$, $x_5 = 3.0000$, $x_6 = 10.0000$, $x_7 = 8.0000$. Consequently, $a(\lambda) = 3\,\lambda^2 + 4\,\lambda^3$, $b(\lambda) = 1 + 2\,\lambda$ and problem L is following:

$$ly = -y'' + \frac{2\,(\pi + x)\,\sin x + 4\,(1 + \cos x)}{(\pi + x + \sin x)^2}\,y = \lambda\,y = s^2\,y,$$

$$\frac{2}{\pi}y(0) + y'(0) + (3\lambda^2 + 4\lambda^3)y(\pi) = 0, \quad (1 + 2\,\lambda)y(0) + \frac{1}{8\pi}y(\pi) + \left(\lambda - \frac{1}{4}\right)y'(\pi) = 0.$$

Acknowledgements This work was supported by the Russian Foundation for Basic Research (project No 15-01-01095 a), by the Academy of Sciences of the Republic of Bashkortostan (project No 14-01-97010-r povolzhe) and by the Scientific and Technological Research Council of Turkey.

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