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BASICITY OF THE SYSTEMS OF SINES AND COSINES WITH LINEAR PHASES IN GRAND-SOBOLEV SPACES

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Abstract. *In this work systems of sines* $1\cup\{\sin(n-\beta)t\}_{n\geq1}$ *and* $1\cup\{\cos(n-\beta)t\}_{n\geq1}$ *cosines are considered, where* β *is real parameter. Criterion for the completeness, minimality and basicity of these systems with respect to the parameter* β *in one subspace of grand-Sobolev space are found.*

Keywords: basicity, trigonometric systems, grand-Lebesgue space, grand-Sobolev space

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1. Introduction

Lately in mathematics, there has been an upsurge of interest in non-standard spaces (see [1], [8]-[10], [15], [16]). The study of differential equations in non-standard Sobolev spaces requires the knowledge of basicity properties of trigonometric systems in corresponding non-standard function spaces. Basicity properties of some trigonometric systems in such spaces have been treated in [6], [7], [11]-[13], [20]. Basicity properties of the systems

$$\{\sin(n-\beta)t\}_{n\geq1}, \tag{1}$$

$$\{\cos(n-\beta)t\}_{n\geq1} \tag{2}$$

in Lebesgue function space are studied in works [17], [18], [21]. This field was further developed by B.T. Bilalov [2]-[5].

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Grand-Lebesgue spaces $L^{p)}$ have been introduced in [10] in the study of Jacobian in an open set. These are the functional Banach spaces, and they have wide applications in the theory of partial differential equations, theory of interpolation, etc. The study of some problems of harmonic analysis in these spaces is of special interest.

As these spaces are not separable, basis and approximation-related problems remained unsolved in them. In [12], some $M^{p)}$ subspace was constructed, interesting from the point of view of the theory of differential equations. In [11], [13], basicity properties of the systems (1) and (2) have been studied in this subspace. Grand-Sobolev spaces have been studied in many works, including [10].

So, let $1 < p < \infty$. A space $L^{p)}(a,b)$ of measurable functions satisfying the condition

$$\|f\|_{p)} = sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{b-a}\int_a^b |f|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} < \infty \tag{3}$$

in the interval $(a,b) \subset R$ is called a grand-Lebesgue space.

Denote by $\tilde{M}^{p)}(a,b)$ the set of all functions satisfying the condition $\left\|\hat{f}(\cdot+\delta)-\hat{f}(\delta)\right\|_{p)} \to 0$ as $\delta \to 0$ and belonging to $L^{p)}(a,b)$, where

$$\hat{f}(t) = \begin{cases} f(t), & t \in (a,b), \\ 0, & t \notin (a,b) \end{cases}.$$

It is clear that the set $\tilde{M}^{p)}(a,b)$ is a manifold in $L^{p)}(a,b)$. Denote by $M^{p)}(a,b)$ the closure of $\tilde{M}^{p)}(a,b)$ with respect to the norm (3).

Denote by ${W_{p)}}^1(a,b)$ the space of functions which belong to $L^{p)}(a,b)$ together with their derivatives equipped with the norm

$$\|f\|_{W_{p)}} = \|f\|_{p)} + \|f'\|_{p)}. \tag{4}$$

We will call this space a grand-Sobolev space:

$${W_{p)}}^1(a,b) = \left\{ f \left| f, f' \in L^{p)}(a,b), \|f\|_{p)} + \|f'\|_{p)} < \infty \right.\right\}.$$

It is easy to prove that this is a Banach space. As is known, $L^{p)}(a,b)$ is not separable. Therefore, ${W_{p)}}^1(a,b)$ is also not a separable space. Denote by $\tilde{M}{W_{p)}}^1(a,b)$ the set of all functions which satisfy the condition $\left\|\hat{f}'(\cdot+\delta)-\hat{f}'(\delta)\right\|_{p)} \to 0$ as $\delta \to 0$ and belong to ${W_{p)}}^1(a,b)$, where

$$\hat{f}(t) = \begin{cases} f(t), & t \in (a,b), \\ 0, & t \notin (a,b) \end{cases}.$$

It is clear that the set $\tilde{M}{W_{p)}}^1(a,b)$ is a manifold in ${W_{p)}}^1(a,b)$. Denote by $M{W_{p)}}^1(a,b)$ the closure of $\tilde{M}{W_{p)}}^1(a,b)$ with respect to the norm (4).

The following lemma is true.

Lemma. *[19] The operator* $A(f,\lambda) = \lambda + \int_a^t f(\tau)\,d\tau$ *creates an isomorphism between the spaces* $M^{p)}(a,b) \bigoplus \mathbb{C}$ *and* ${MW_{p)}}^1(a,b)$*, where* $\mathbb{C}$ *is a complex plane,* $1 < p < \infty$.

Theorem 1. *Let* $2\beta + \frac{1}{p} \notin \mathbb{Z}$*,* $\beta \neq 1$*,* $1 < p < \infty$ *Then the system*

$$1 \cup \{\sin(n-\beta)\,t\}_{n\geq 1}$$

forms a basis for the space $MW^1_{p)}(0,\pi)$*,* $1 < p < \infty$*, if and only if* $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] = 0$*. Moreover, for* $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] < 0$ *it is not complete, but is minimal; and for* $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] > 0$ *it is complete, but is not minimal in* $MW^1_{p)}(0,\pi)$.

Proof. It is known that system $\{\cos(n-\beta)\,t\}_{n\geq 1}$ in a basis a spaces $M^{p)}(0,\pi)$, if $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] = 0$ [13]. Let's first prove that the system $\hat{u}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\hat{u}_n = \begin{pmatrix} (n-\beta)\cos(n-\beta)t \\ 0 \end{pmatrix}$, $n \geq 1$ forms a basis for the space $M^{p)}(0,\pi) \oplus \mathbb{C}$. To do so, it suffices to show that $\forall \hat{u} = \begin{pmatrix} u \\ \lambda \end{pmatrix} \in M^{p)}(0,\pi) \oplus \mathbb{C}$ the expansion

$$\hat{u} = \sum_{n=0}^{\infty} c_n \hat{u}_n \tag{5}$$

exists and is unique. This expansion is equivalent to following expansion

$$u(t) = \sum_{n=1}^{\infty} c_n (n-\beta)\cos(n-\beta)t \tag{6}$$

and equality $\lambda = c_0$.

Following [13] we obtain that there exists the expansion (6) exists and it is unique. Therefore, the expansion (5) also exists and is unique, i.e. the system $\{\hat{u}_n\}_{n\geq 0}$ forms a basis for the space $M^{p)}(0,\pi) \oplus \mathbb{C}$. As the operator A is an isomorphism, the system $\{A\hat{u}_n\}_{n\geq 0}$ must form a basis for the space $MW^1_{p)}(0,\pi)$. Simple calculations show that

$$A\hat{u}_0 = 1,$$

$$A\hat{u}_n = \sin(n-\beta)t, n \geq 1.$$

That is, the system $1 \cup \{\sin(n-\beta)\,t\}_{n\geq 1}$ forms a basis for the space $MW^1_{p)}(0,\pi)$ if, $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] = 0$.

Consider the case when $\left[\beta + \frac{1}{2p} - \frac{1}{2}\right] < 0$. Let, for example, the inequalities

$$-1 < \beta + \frac{1}{2p} - \frac{1}{2} < 0 \Leftrightarrow 0 < (\beta+1) + \frac{1}{2p} - \frac{1}{2} < 1$$

is holds. Consider the system $1 \cup \{\sin(n-\beta)t\}_{n\geq 0}$ and transform it

$$1 \cup \{\sin(n+1-(\beta+1))t\}_{n\geq 0} \equiv 1 \cup \{\sin(n-\beta_1)t\}_{n\geq 1},$$

where $\beta_1 = \beta + 1$. Consequently, the following inequality

$$0 < \beta_1 + \frac{1}{2p} - \frac{1}{2} < 1$$

hold and as a result the system $1 \cup \{\sin(n-\beta)t\}_{n\geq 0}$ form a basis, and there- fore the system $1 \cup \{\sin(n-\beta)t\}_{n\geq 1}$ is minimal, but is not complete in $MW^1_{p)}(0,\pi)$. Continuing this process, we obtain for $\beta + \frac{1}{2p} - \frac{1}{2} < 0$ the system $1 \cup \{\sin(n-\beta)t\}_{n\geq 1}$ is minimal, but is not complete in $MW^1_{p)}(0,\pi)$.

With similar reasoning, it is proved that for $\beta + \frac{1}{2p} - \frac{1}{2} > 1$, the system $1 \cup \{\sin(n-\beta)t\}_{n\geq 1}$ is complete, but is not minimal in $MW^1_{p)}(0,\pi)$. So, the theorem is proved. ◀

Remark. If $\beta = 1$, then it can be shown in a similar way that $1 \cup \{t\} \cup \{\sin(nt)\}_{n\geq 1}$ is a basis in $MW^1_{p)}(0,\pi)$.

Theorem 2. *Let $2\beta + \frac{1}{p} \notin Z$, $1 < p < \infty$. The system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ forms a basis for $MW^1_{p)}(0,\pi)$ if and only if $\left[\beta + \frac{1}{2p}\right] = 0$ Moreover, for $\left[\beta + \frac{1}{2p}\right] < 0$ it is not complete, but is minimal; and for $\left[\beta + \frac{1}{2p}\right] > 0$ it is complete, but is not minimal in $MW^1_{p)}(0,\pi)$.*

Proof. It is know a system $\{\sin(n-\beta)t\}_{n\geq 1}$ is a basis in space $MW^1_{p)}(0,\pi)$, if $\left[\beta + \frac{1}{2p}\right] = 0$ [13]. We will prove that the system $\{\hat{u}_n\}_{n\geq 0}$ forms a basis for $M^{p)}(0,\pi) \oplus \mathbb{C}$, where

$$\hat{u}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \hat{u}_n = \begin{pmatrix} -(n-\beta)\sin(n-\beta)t \\ 1 \end{pmatrix}, n \geq 1.$$

Let us show that for $\forall \hat{u} = \begin{pmatrix} u \\ \lambda \end{pmatrix} \in M^{p)}(0,\pi) \oplus \mathbb{C}$ there exists decomposition

$$\hat{u} = \sum_{n=0}^{\infty} c_n \hat{u}_n \tag{7}$$

and this decomposition is unique.

This decomposition is equivalent to the next two decompositions

$$u(t) = -\sum_{n=1}^{\infty} c_n (n-\beta)\sin(n-\beta)t, \tag{8}$$

$$\lambda = \sum_{n=0}^{\infty} c_n. \tag{9}$$

Following [13] we obtain that (8) uniquely exists and belongs to the space $M^{p)}(0,\pi)$. Since $\forall\varepsilon \in (0, p-1)$ $L^{p)} \subset L^{p-\varepsilon}$ then, by using the main result of [14] if $1 < p \leq 2$, $\beta + \frac{1}{2p} < 1$, then the Hausdorf-Young inequality holds for the system $\{\sin(n-\beta)t\}_{n\geq 1}$. That is for $1 < p \leq 2$, we have

$$\left(\sum_{n=1}^{\infty} |c_n(n-\beta)|^q\right)^{\frac{1}{q}} < M\, \|u\|_{p-\varepsilon}\,,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Using Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} |c_n(n-\beta)| \frac{1}{|n-\beta|} \leq \left(\sum_{n=1}^{\infty} |c_n(n-\beta)|^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} \frac{1}{|n-\beta|^p}\right)^{\frac{1}{p}} < +\infty.$$

When $2 < p$, we can find $\varepsilon > 0$ such that $2 < p - \varepsilon$. Therefore,

$$L^{p)} \subset L^{p-\varepsilon} \subset L^2.$$

Similarly we have

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} |c_n(n-\beta)| \frac{1}{|n-\beta|} \leq \left(\sum_{n=1}^{\infty} |c_n(n-\beta)|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{|n-\beta|^2}\right)^{\frac{1}{2}} < +\infty.$$

So, the series $\sum\limits_{n=1}^{\infty} |c_n|$ is convergent. Therefore, the expansion (9) also exists and is unique. This implies the existence and uniqueness of the expansion (7), i.e. the system

$$\{\hat{u}_n\}_{n\geq 0}$$

forms a basis for the space $M^{p)}(0,\pi) \oplus \mathbb{C}$. As the operator A is an isomorphism, the system

$$\{A\hat{u}_n\}_{n\geq 0}\,,$$

must form a basis for the space $MW^1_{p)}(0,\pi)$. Simple calculations show that

$$A\hat{u}_0 = 1,$$

$$A\hat{u}_n = \cos(n-\beta)\,t, \quad n \geq 1.$$

That is, the system $1 \cup \{\cos(n-\beta)\,t\}_{n\geq 1}$ forms a basis for the space $MW^1_{p)}(0,\pi)$ if, $\left[\beta + \frac{1}{2p}\right] = 0$.

Consider the case when $\left[\beta + \frac{1}{2p}\right] < 0$. Let, for example,the inequalities

$$-1 < \beta + \frac{1}{2p} < 0 \Leftrightarrow 0 < (\beta+1) + \frac{1}{2p} < 1$$

is holds. Consider the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 0}$ and transform it

$$1 \cup \{\cos(n+1-(\beta+1))t\}_{n\geq 0} \equiv 1 \cup \{\cos(n-\beta_1)t\}_{n\geq 1},$$

where $\beta_1 = \beta + 1$. Consequently, the following inequality

$$0 < \beta_1 + \frac{1}{2p} < 1$$

hold and as a result the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 0}$ form a basis, and there- fore the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ is minimal, but is not complete in $MW^1_{p)}(0,\pi)$. Continuing this process, we obtain for $\beta + \frac{1}{2p} < 0$ the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ is minimal, but is not complete in $MW^1_{p)}(0,\pi)$.

With similar reasoning, it is proved that for $\beta + \frac{1}{2p} > 1$, the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ is complete, but is not minimal in $MW^1_{p)}(0,\pi)$. So, the theorem is proved. ◄

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