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# THE REGULARIZED TRACE FORMULA FOR "WEIGHTED" STURM-LIOUVILLE EQUATION WITH $\delta$-INTERACTION

**A. KARAKUŞ, M.Dzh. MANAFOV***

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**Abstract.** *The regularized trace formula of first order for the "weighted" Sturm-Liouville equation with point $\delta$-interaction is obtained.*

**Keywords**: "weighted" Sturm-Liouville equation, regularized trace formula, point $\delta$-interaction

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## 1. Introduction

In a finite dimensional space, an operator has a finite trace. It is well known that it is easy to compute the sum of the eigenvalues for a matrix, trace of the matrix, i.e. to the sum of the elements in the principal diagonal.

However, in an infinite-dimensional space, in general, ordinary differential operators do not have a finite trace. In 1953, Gel'fand and Levitan [7], assuming the continuous differentiability of the function $q(x)$, obtained the following remarkable formula for the regularized trace:

$$\sum_{n=0}^{\infty}[\lambda_n - n^2 - \frac{1}{\pi}\int_0^{\pi} q(x)dx] = \frac{q(0)+q(\pi)}{4} - \frac{1}{2\pi}\int_0^{\pi} q(x)dx.$$

Here, $\lambda_n$ are eigenvalues of the operator

$$-y'' + q(x) = \lambda y, \quad y'(0) = y'(\pi) = 0.$$

* Corresponding author.

**Abdullah Karakuş**
Adıyaman University, Adıyaman, Türkiye
E-mail: abdullahkarakus02@hotmail.com

**Manaf Dzh. Manafov**
Adıyaman University, Adıyaman, Türkiye;
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: mmanafov@adiyaman.edu.tr

This formula gave rise to a large and very important theory, which started from the investigation of specific operators and further embraced the analysis of regularized traces of discrete operators in general form. In a short time, a number of authors turned their attention to trace theory and obtained interesting results. This work was continued by many authors. The current situation of this subject and studies related to it are presented in the comprehensive survey paper [15]. The reqularized trace for differential equations are found in [5]–[11]. However, there is a small numbers of words on the regularized trace for Sturm-Liouville operators with singular potentials (see [2], [13], [15]–[18]). The trace identity of a differential operator deeply reveals spectral structure of the differential operator and has important applications in the numerical calculation of eigenvalues, inverse problem, theory of solutions, and theory of integrable system (see [12], [14]).

We consider the boundary value problem for the differential equation

$$\ell y := \frac{1}{r(x)}[-y'' + q(x)y] = \lambda y, \quad x \in (0,a) \cup (a,\pi), \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) = 0, \quad V(y) := y(\pi) = 0 \tag{2}$$

and conditions at the point $x = a$,

$$I(y) := \begin{cases} y(a+0) = y(a-0) \equiv y(a), \\ y'(a+0) - y'(a-0) = -\alpha\lambda y(a), \end{cases} \tag{3}$$

where $q(x)$ is real-valued function in $W_2^1(0,\pi)$ and $\alpha > 0$, $\lambda$ is spectral parameter, $r(x)$ is piecewise constant function:

$$r(x) = \begin{cases} r^2, & 0 \le x < a, \\ 1, & a \le x \le \pi, \quad 0 < r \ne 1. \end{cases}$$

Notice that, we can understand (1),(3) as studying the equation

$$-y'' + q(x)y = \lambda\rho(x)y, \quad x \in (0,\pi), \tag{4}$$

when $\rho(x) = r(x) + \alpha\delta(x-a)$, where $\delta(x)$ is the Dirac function (see [1]).

In the present paper, after construction of the Hilbert space related to (4), we obtain the formula of the first order reqularized trace for "weighted" Sturm-Liouville equation with point $\delta$-interaction.

## 2. Construction of the Hilbert Space Related of the Problem and Some Properties of Its Spectral Characteristics

Problem (1)-(3) reduced to the eigenvalue problem for the linear operator $L$ in the Hilbert space $\mathcal{H} = L_2(0,\pi) \oplus \mathbb{C}$ with inner product

$$\langle f, g\rangle_{\mathcal{H}} := \langle f_1, g_1\rangle_{L_2} + \alpha^{-1} f_2 \bar{g}_2$$

for
$$f = \begin{pmatrix} f_1(x) \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1(x) \\ g_2 \end{pmatrix}.$$

Consider the operator
$$Lf := \begin{pmatrix} \ell f_1 \\ f_1'(a-0) - f_1'(a+0) \end{pmatrix}$$

with the domain
$$D(L) := \{f \in \mathcal{H} | f_1, f_1' \in AC[(0,\pi) \setminus \{a\}], \ell f_1 \in L_2[(0,\pi) \setminus \{a\}], f_2 = \alpha f_1(a),$$
$$U(f_1) = V(f_1) = 0\}.$$

Here $AC(\cdot)$ denotes the set of all absolutely continuous functions on related interval. In particular, those functions have limits at the point $a$.

**Theorem 1.** *The operator $L$ is symmetric.*

*Proof.* Since $f$ and $\bar{g}$ satisfy the same boundary conditions (2) and from the conditions at the point $x = a$, we obtain $\langle Lf, g\rangle_{\mathcal{H}} = \langle f, Lg\rangle_{\mathcal{H}}$ for $f, g \in D(L)$. So $L$ is symmetric. ◀

**Corollary.** *The function $W(f, g; x) = f(x)g'(x) - f'(x)g(x)$ is contionus on $(0, \pi)$.*

Let $\varphi(x,\lambda), \psi(x,\lambda), C(x,\lambda), S(x,\lambda)$ be solutions of (1) under the initial conditions
$$C(0,\lambda) = S'(0,\lambda) = \varphi(0,\lambda) = \psi(\pi,\lambda) = 1,$$
$$C'(0,\lambda) = S(0,\lambda) = \varphi'(0,\lambda) = \psi'(\pi,\lambda) = 0,$$

and under the conditions (3).

Clearly,
$$U(\varphi) = V(\psi) = 0.$$

Denote
$$\triangle(\lambda) = W(\varphi, \psi; x). \tag{5}$$

By virtue of Corollary and Ostrogradskii-Liouville theorem (see [3, p. 83]), $\triangle(\lambda)$ does not depend on $x$. The function $\triangle(\lambda)$ is called characteristic function of $L$. Substituting $x = 0$ and $x = \pi$ into (5), we get
$$\triangle(\lambda) = V(\varphi) = U(\psi). \tag{6}$$

The function $\triangle(\lambda)$ is entire in $\lambda$, and it has an at most countable set of zeros $\{\lambda_n\}_{n=1,2,3...}$.

Now, consider the solution $\varphi(x,\lambda)$. Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ be smooth solutions of (1) on the interval $(0,\pi)$ under the initial conditions $C_0(0,\lambda) = S_0'(0,\lambda) = 1, C_0'(0,\lambda) = S_0(0,\lambda) = 0$. Then
$$C(x,\lambda) = C_0(x, r\sqrt{\lambda}), S(x,\lambda) = S_0(x, r\sqrt{\lambda}), \quad x < a, \tag{7}$$

$$
\begin{aligned}
C(x,\lambda) &= A_1 C_0(x,\sqrt{\lambda}) + B_1 S_0(x,\sqrt{\lambda}),\\
S(x,\lambda) &= A_2 C_0(x,\sqrt{\lambda}) + B_2 S_0(x,\sqrt{\lambda}), \quad x>a,
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
A_1 &= C_0(a,r\sqrt{\lambda})S_0'(a,\sqrt{\lambda}) - C_0'(a,r\sqrt{\lambda})S_0(a,\sqrt{\lambda}) + \alpha k^2 C_0(a,r\sqrt{\lambda})S_0(a,\sqrt{\lambda}),\\
B_1 &= C_0(a,\sqrt{\lambda})C_0'(a,r\sqrt{\lambda}) - C_0'(a,\sqrt{\lambda})C_0(a,r\sqrt{\lambda}) - \alpha k^2 C_0(a,r\sqrt{\lambda})C_0(a,\sqrt{\lambda}),\\
A_2 &= S_0(a,r\sqrt{\lambda})S_0'(a,\sqrt{\lambda}) - S_0'(a,r\sqrt{\lambda})S_0(a,\sqrt{\lambda}) + \alpha k^2 S_0(a,r\sqrt{\lambda})S_0(a,\sqrt{\lambda}),\\
B_2 &= C_0(a,\sqrt{\lambda})S_0'(a,r\sqrt{\lambda}) - C_0'(a,\sqrt{\lambda})S_0(a,r\sqrt{\lambda}) - \alpha k^2 C_0(a,\sqrt{\lambda})S_0(a,r\sqrt{\lambda}).
\end{aligned} \tag{9}
$$

It is easy to verify that function $C_0(x,\lambda)$ satisfies the following integral equation:

$$
C_0(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} g(t) C_0(t,\lambda)dt. \tag{10}
$$

Solving the equation (10) by the method of successive approximations, we obtain

$$
\begin{aligned}
C_0(x,\lambda) = \cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}\int_0^x q(t)dt + \frac{\cos\sqrt{\lambda}x}{4\lambda}\Bigg\{ q(x) - q(0) -\\
-\frac{1}{2}\left[\int_0^x q(t)dt\right]^2\Bigg\} + O\left(\frac{1}{\lambda^{3/2}}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right| x\right)\right),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
C_0'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \frac{\cos\sqrt{\lambda}x}{2}\int_0^x q(t)dt + \frac{\sin\sqrt{\lambda}x}{4\lambda}\Bigg\{ q(x) + q(0) +\\
+\frac{1}{2}\left[\int_0^x q(t)dt\right]^2\Bigg\} + O\left(\frac{1}{\lambda}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right| x\right)\right).
\end{aligned} \tag{12}
$$

Analogously,

$$
\begin{aligned}
S_0(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}x}{2\lambda}\int_0^x q(t)dt + \frac{\sin\sqrt{\lambda}x}{4\lambda^{3/2}}\Bigg\{ q(x) + q(0) -\\
-\frac{1}{2}\left[\int_0^x q(t)dt\right]^2\Bigg\} + O\left(\frac{1}{\lambda^2}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right| x\right)\right),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
S_0'(x,\lambda) = \cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}\int_0^x q(t)dt + \frac{\cos\sqrt{\lambda}x}{4\lambda}\Bigg\{ -q(x) + q(0) -\\
-\frac{1}{2}\left[\int_0^x q(t)dt\right]^2\Bigg\} + O\left(\frac{1}{\lambda^{3/2}}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right| x\right)\right).
\end{aligned} \tag{14}
$$

By virtue of (9) and (11)-(14)

$$
A_1 = \alpha\sqrt{\lambda}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}a + \cos\sqrt{\lambda}ra\cos\sqrt{\lambda}a\left[1 - \frac{\alpha}{2}\int_0^a q(t)dt\right] +
$$

$$+\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}a\left[r+\frac{\alpha}{2r}\int_0^a q(t)dt\right]+$$

$$+\frac{1}{2\sqrt{\lambda}}\sin\sqrt{\lambda}ra\cos\sqrt{\lambda}a\left[\left(\frac{1}{r}-r\right)\int_0^a q(t)dt-\frac{\alpha}{2r}\left(\int_0^a q(t)dt\right)^2\right]+$$

$$+\frac{\alpha}{4\sqrt{\lambda}}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}a\left[\left(1+\frac{1}{r^2}\right)q(a)+\left(1-\frac{1}{r^2}\right)q(0)-\right.$$

$$\left.-\frac{1}{2}\left(1+\frac{1}{r^2}\right)\left(\int_0^a q(t)dt\right)^2\right]+O\left(\frac{1}{\lambda}\right),$$

$$B_1=-\alpha\lambda\cos\sqrt{\lambda}ra\cos\sqrt{\lambda}a-\sqrt{\lambda}\sin\sqrt{\lambda}ra\cos\sqrt{\lambda}a\left[r+\frac{\alpha}{2r}\int_0^a q(t)dt\right]+$$

$$+\sqrt{\lambda}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}a\left[1-\frac{\alpha}{2}\int_0^a q(t)dt\right]+$$

$$+\frac{1}{2}\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}a\left[\left(\frac{1}{r}-r\right)\int_0^a q(t)dt-\frac{\alpha}{2r}\left(\int_0^a q(t)dt\right)^2\right]+$$

$$+\frac{\alpha}{4}\cos\sqrt{\lambda}ra\cos\sqrt{\lambda}a\left(1+\frac{1}{r^2}\right)\left[q(a)-q(0)-\frac{1}{2}\left(\int_0^a q(t)dt\right)^2\right]+O\left(\frac{1}{\lambda^{\frac{1}{2}}}\right),$$

$$A_2=\frac{\alpha}{r}\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}a+\frac{1}{r\sqrt{\lambda}}\sin\sqrt{\lambda}ra\cos\sqrt{\lambda}a\left[1-\frac{\alpha}{2}\int_0^a q(t)dt\right]-$$

$$-\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}a[1+\frac{\alpha}{2r^2}\int_0^a q(t)dt\Big]+O\left(\frac{1}{\lambda}\right),$$

$$B_2=-\frac{\alpha}{r}\sqrt{\lambda}\sin\sqrt{\lambda}ra\cos\sqrt{\lambda}a+\cos\sqrt{\lambda}ra\cos\sqrt{\lambda}a[1+\frac{\alpha}{2r^2}\int_0^a q(t)dt\Big]+$$

$$+\frac{1}{r}\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}a\left[1-\frac{\alpha}{2}\int_0^a q(t)dt\right]+O\left(\frac{1}{\lambda^{\frac{1}{2}}}\right).$$

Since $\varphi(x,\lambda)=C(x,\lambda)$, we calculate using (7)-(14)

$$\varphi(x,\lambda)=\cos\sqrt{\lambda}rx+\frac{\sin\sqrt{\lambda}rx}{2\sqrt{\lambda}r}\int_0^x q(t)dt+\frac{\cos\sqrt{\lambda}rx}{4\lambda r^2}\left\{q(x)-q(0)-\frac{1}{2}\left[\int_0^x q(t)dt\right]^2\right\}+$$

$$+O\left(\frac{1}{\lambda^{3/2}}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right|rx\right)\right),\quad x<a,$$

$$\varphi(x,\lambda)=-\alpha\sqrt{\lambda}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}(x-a)+\cos\sqrt{\lambda}ra\cos\sqrt{\lambda}(x-a)\left[1+\frac{\alpha}{2}\int_a^x q(t)dt\right]-$$

$$-\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}(x-a)\left[r+\frac{\alpha}{2r}\int_0^a q(t)dt\right]+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}ra\cos\sqrt{\lambda}(x-a)\times$$

$$\times\left[\frac{1}{2r}\int_0^a q(t)dt+\frac{r}{2}\int_a^x q(t)dt+\frac{\alpha}{4r}\int_0^a q(t)dt\int_a^x q(t)dt\right]+$$

$$+\frac{\alpha}{\sqrt{\lambda}}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}(x-a)\times\left\{-\frac{q(x)+q(0)}{4}+\frac{1}{8}\left[\int_0^x q(t)dt\right]^2+\right.$$

$$\left.+\frac{1}{2\alpha}\int_0^x q(t)dt-\frac{1}{4}\int_0^a q(t)dt\int_0^x q(t)dt\right\}+\frac{\alpha}{\sqrt{\lambda}}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}(x+a)\times$$

$$\times\left\{\frac{1}{4}\left(1+\frac{1}{r^2}\right)\left[q(a)-q(0)-\frac{1}{2}\left(\int_0^a q(t)dt\right)^2\right]\right\}+$$

$$+O\left(\frac{1}{\lambda}\exp\left(\left|\mathrm{Im}\sqrt{\lambda}\right|x\right)\right),\quad x>a. \tag{15}$$

It follows from (6) and (15) that at $\mathrm{Im}\sqrt{\lambda}\to+\infty$

$$\triangle(\lambda)=\varphi(\pi,\lambda)=\left[C_0\sqrt{\lambda}+\frac{1}{4}(1+r)+C_1+C_2\frac{1}{\sqrt{\lambda}}\right]e^{-i\sqrt{\lambda}ra}\cdot e^{-i\sqrt{\lambda}(\pi-a)}+$$

$$+O\left(\frac{1}{\lambda}e^{-i\sqrt{\lambda}ra}\cdot e^{-i\sqrt{\lambda}(\pi-a)}\right),$$

where

$$C_0=\frac{\alpha}{4i},\quad C_1=\frac{1}{8}\left[\frac{\alpha}{r}\int_0^a q(t)dt+\alpha\int_a^\pi q(t)dt\right],$$

$$C_2=\frac{1}{4i}\left\{\frac{\alpha}{4}[q(\pi)+q(0)]-\frac{1}{2}\int_0^\pi q(t)dt+\frac{1}{2r}\int_0^a q(t)dt+\frac{r}{2}\int_a^\pi q(t)dt+\right.$$

$$\left.+\frac{\alpha}{4}\int_0^a q(t)dt\int_0^\pi q(t)dt+\frac{\alpha}{4r}\int_0^a q(t)dt\int_a^\pi q(t)dt-\frac{\alpha}{8}\left[\int_0^\pi q(t)dt\right]^2\right\}.$$

Denote by $\varphi_0(x,\lambda)$ the solution of equation (1) for $q(x)=0$, that satisfies the initial conditions $\varphi_0(0,\lambda)=1,\varphi_0'(0,\lambda)=0$. Then it is obvious that the eigenvalues of the boundary value problem (1)-(3) for $q(x)=0$ coincide with the roots of the entire function $\varphi_0(\pi,\lambda)$:

$$\varphi_0(x,\lambda)=-\alpha\sqrt{\lambda}\cos\sqrt{\lambda}ra\sin\sqrt{\lambda}(x-a)+\cos\sqrt{\lambda}ra\cos\sqrt{\lambda}(x-a)-$$

$$-r\sin\sqrt{\lambda}ra\sin\sqrt{\lambda}(x-a).$$

Equation $\varphi_0(\pi,\lambda)=0$ takes the form

$$\alpha\sqrt{\lambda}\tan\sqrt{\lambda}(\pi-a)+r\tan\sqrt{\lambda}ra\cdot\tan\sqrt{\lambda}(\pi-a)=1. \tag{16}$$

The roots of the function $\varphi_0(\pi,\lambda)$, and hence the eigenvalues $\lambda_1^0,\lambda_2^0,\cdots,\lambda_n^0,\cdots$ are determined from the equation (16). Then

$$\varphi(\pi,\lambda)=\varphi_0(\pi,\lambda)+\left(C_1+\frac{1}{\sqrt{\lambda}}C_2\right)e^{-i\sqrt{\lambda}ra}\cdot e^{-i\sqrt{\lambda}(\pi-a)}+O\left(\frac{1}{\lambda}\right).$$

# 3. Trace of the Problem

The goal of this paper obtained some formulas for the regularized traces of first order for the problem (1)-(3).

**Theorem 2.** *Suppose that $g(x) \in W_2^1(0,\pi)$ and*

$$\frac{\alpha}{r}\int_0^a q(t)dt + \alpha\int_a^\pi q(t)dt = 0.$$

*Then it holds*

$$\sum_{n=1}^{\infty}(\lambda_n - \lambda_n^0) = \frac{1}{4}[q(0)+q(\pi)] + \frac{1}{2\alpha}\left(\frac{1}{r}-1\right)\int_0^a q(t)dt +$$
$$+\frac{1}{2\alpha}(r-1)\int_a^\pi q(t)dt + \frac{1}{4}\int_0^a q(t)dt\int_0^\pi q(t)dt +$$
$$+\frac{1}{4r}\int_0^a q(t)dt\int_a^\pi q(t)dt - \frac{1}{8}\left[\int_0^\pi q(t)dt\right]^2.$$

*Proof.* By the residue theorem (see [4], p. 125) for $R \to +\infty$ we have

$$\sum_{|\lambda_n|<R}(\lambda_n - \lambda_n^0) = \frac{1}{2\pi i}\oint_{|\lambda|=R}\lambda\left[\frac{\frac{d}{d\lambda}\varphi(\pi,\lambda)}{\varphi(\pi,\lambda)} - \frac{\frac{d}{d\lambda}\varphi_0(\pi,\lambda)}{\varphi_0(\pi,\lambda)}\right]d\lambda =$$
$$= -\frac{1}{2\pi i}\frac{C_1}{C_0}\oint_{|\lambda|=R}\frac{d\lambda}{\sqrt{\lambda}} - \frac{1}{2\pi i}\frac{C_2}{C_0}\oint_{|\lambda|=R}\frac{d\lambda}{\lambda} - \frac{1}{2\pi i}\oint_{|\lambda|=R}O\left(\frac{1}{\lambda}\right)d\lambda.$$

Given that

$$\oint_{|\lambda|=R}\frac{d\lambda}{\sqrt{\lambda}} = -4\sqrt{R}, \quad \oint_{|\lambda|=R}\frac{d\lambda}{\lambda} = 2\pi i, \quad \left|\oint_{|\lambda|=R}O\left(\frac{1}{\lambda}\right)d\lambda\right| < O(1),$$

we get

$$\sum_{|\lambda_n|<R}(\lambda_n - \lambda_n^0) = \frac{4\sqrt{R}}{2\pi i}\frac{C_1}{C_0} - \frac{2\pi i}{2\pi i}\frac{C_2}{C_0} + O(1).$$

Substituting the values here $C_0, C_1, C_2$ for $R \to +\infty$ we obtain

$$\sum_{n=1}^{\infty}(\lambda_n - \lambda_n^0) = \frac{1}{4}[q(0)+q(\pi)] + \frac{1}{2\alpha}\left(\frac{1}{r}-1\right)\int_0^a q(t)dt +$$
$$+\frac{1}{2\alpha}(r-1)\int_a^\pi q(t)dt + \frac{1}{4}\int_0^a q(t)dt\int_0^\pi q(t)dt +$$
$$+\frac{1}{4r}\int_0^a q(t)dt\int_a^\pi q(t)dt - \frac{1}{8}\left[\int_0^\pi q(t)dt\right]^2,$$

completing the proof of theorem. ◄

# References


1. Albeverio S., Gesztesy F., Høegh-Krohn R., Holden H. *Solvable Models in Quantum Mechanics.* Amer. Math. Soc., Providence, RI, 2005.
2. Aliev A.R., Manafov M.Dzh. Trace formula for a Sturm-Liouville operator with a $\delta'$-interaction point. *Differ. Equ.*, 2021, **57** (5), pp. 563–569.
3. Coddington E.A., Levinson N. *Theory of Ordinary Differential Equations.* McGraw-Hill, New York, 1955.
4. Conway J.B. *Functions of One Complex Variable II.* Springer, New York, 1995.
5. Dikii L.A. Trace formulas for Sturm-Liouville differential operators. *Uspekhi Mat. Nauk*, 1958, **13** (3(81)), pp. 111–143 (in Russian).
6. Gasymov M.G. On the sum of the differences of the eigenvalues of two self-adjoint operators. *Soviet Math. Dokl.*, 1963, **4**, pp. 838–842.
7. Gel'fand I.M., Levitan B.M. A simple identity for the eigenvalues of a second-order differential operator. *Dokl. Akad. Nauk SSSR*, 1953, **88** (4), pp. 593–596 (in Russian).
8. Gesztesy F., Holden H., Simon B., Zhao Z. A trace formula for multidimensional Schrödinger operators. *J. Funct. Anal.*, 1996, **141** (2), pp. 449–465.
9. Guseinov G.Sh., Levitan B.M. On trace formulas for Sturm-Liouville operators. *Vestn. Mosk. Univ. Ser. Mat. Mekh.*, 1978, (1), pp. 40–49 (in Russian).
10. Lax P.D. Trace formulas for the Schroedinger operator. *Commun. Pure Appl. Math.*, 1994, **47** (4), pp. 503–512.
11. Levitan B.M. Calculation of the regularized trace for the Sturm-Liouville operator. *Uspekhi Mat. Nauk*, 1964, **19** (1(115)), pp. 161–165 (in Russian).
12. Levitan B.M. *Inverse Sturm-Liouville Problems.* Nauka, Moscow, 1984; VSP, Zeist, 1987.
13. Manafov M.Dzh. A regularized trace formula for "weighted" Sturm-Liouville equation with point $\delta$-interaction. *Turkish J. Math.*, 2021, **45** (4), pp. 1767–1774.
14. Marchenko V.A. *Sturm-Liouville Operators and Applications.* Birkhäuser, Basel, 1986.
15. Sadovnichii V.A., Podolskii V.E. Traces of operators. *Russ. Math. Surv.*, 2006, **61** (5), pp. 885–953.
16. Savchuk A.M. First-order regularised trace of the Sturm-Liouville operator with $\delta$-potential. *Russ. Math. Surv.*, 2000, **55** (6), pp. 1168–1169.
17. Savchuk A.M., Shkalikov A.A. Trace formula for Sturm-Liouville operators with singular potentials. *Math. Notes*, 2001, **69** (3), pp. 387–400.
18. Vinokurov V.A., Sadovnichii V.A. The asymptotics of eigenvalues and eigenfunctions and a trace formula for a potential with delta functions. *Differ. Equ.*, 2002, **38** (6), pp. 772–789.