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STUDYING THE SOLUTION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OBTAINED IN SIMULATION OF GAS-LIFT PROCESS BY THE RELAXATION METHOD

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Abstract. *Using the relation method for the system of first order, two-dimensional hyperbolic type partial differential equations describing the motion of gas liquid gas mixture corresponding to gas-lift process of oil production in an annular and lifting pipe, we consider a boundary condition problem. In this problem the existence a boundary condition problem. In this problem the existence of the solution with respect to the equations of motion is studied and it is shown that when constructing the solution of this problem by means of boundary conditions, it is impossible to determine the coefficients of positive degrees of the parameter ε. For this reason, the solution is sought in the form of a series using negative degrees of the parameter ε.*

Keywords: gas-lift, hyperbolic equation,differential equation,relaxation method,integral equation

Mathematics Subject Classification (2020): 35F20, 65M25

1. Introduction

As is known [9] the first method for operation of oil-wells is the fountain method. This time oil rises to the surface due to internal energy of the layer. After some period, the layers energy decreases and gas production stops. As the end of the fountain method, for restoring this method a compressed gas is injected to the well [3], [13]. The gas lightness the oil in the layer and a result the oil emerges to surface. The installations that work using such natural gas are called gas-lifts. One of the important stages of oil production

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is the gas-lift method. Various mathematical models describing the motion in gas-lift process were worked out [1], [2], [8] and different problems were stated by their means, for example maximum oil recovery by injecting minimum gas [4], [5], determination of the hydraulic resistance factor and so in [7]-[9]. In the paper we consider the case when the equations of motion contain a small parameter, and this small parameter is the inverse value of the well depth. The existence of the solution with respect to the equations of motion is studied [11], [12] and it is shown that when building the solution of this problem by means of boundary conditions it is impossible to determine the coefficient of positive degrees of the parameter ε. For this reason, the solution is sought in the form of a series using the negative degrees of the parameter ε [10].

2. Problem Statement and Basic Results

It is known that [1], [6], [8], during the gas-lift process the system of hyperbolic partial differential equations characterizing the motion of liquid-gas mixture in the gas and lifting pipe in annular space is as follows:

$$\begin{cases} \dfrac{\partial P_i\left(x,t\right)}{\partial t} = -\dfrac{c_i^2}{F_i}\cdot\dfrac{\partial Q_i\left(x,t\right)}{\partial x}, & i=1,2, \\ \dfrac{\partial Q_i\left(x,t\right)}{\partial t} = -F_i\dfrac{\partial P_i\left(x,t\right)}{\partial x} - 2a_iQ_i\left(x,t\right), & t>0, \quad x\in\left(0,2l\right). \end{cases} \tag{1}$$

Here $P_i\left(x,t\right)$-is gas pressure injected to the well (liquid-gas mixture in na lifting pipe), $Q_i\left(x,t\right)$ is gas volume, c_i is sound speed, l is well depth, the parameter a_i is found by means of the expression $2a_i = \dfrac{g}{\omega_c} + \dfrac{\lambda\omega}{2D}$. In this expression λ is hydraulic resistance factor, g is free fall acceleration, D is an effective diameter of the annular space and of the hoist. This indices 1 and 2 are the parameters describing the motion in the annular space and lifting pipe, respectively [3], [6].

If we want to solve the system of equations (1) by applying the straightline method and determine the volume and pressure of liquid-gas mixture at each point, this time the umber of equations in the system of differential equations will be too large and this will cause serious errors in computer calculation. Therefore, introducing the parameterεaccording to the relaxation method, we consider the solution of the problem.

I.e. in the system (1) to the inverse value of the well depth as a gas small parameter we make the substation $\varepsilon = \dfrac{1}{2l}$ and $z = \dfrac{x}{2l} = \varepsilon x$ [11]. As a result from the system (1) we get the system of equations:

$$\begin{cases} \dfrac{\partial P_i\left(z,t,\varepsilon\right)}{\partial t} = -\dfrac{c_i^2}{F_i}\cdot\dfrac{\partial Q_i\left(z,t,\varepsilon\right)}{\partial z}\varepsilon, \\ \dfrac{\partial Q_i\left(z,t,\varepsilon\right)}{\partial t} = -F_i\dfrac{\partial P_i\left(z,t,\varepsilon\right)}{\partial z}\varepsilon - 2a_iQ_i\left(z,t,\varepsilon\right). \end{cases} \tag{2}$$

We study the solution of the obtained system of equations (2) within the following boundary condition [2], [3], [12]:

$$\begin{cases} P\left(0,t,\varepsilon\right) = P^0\left(t,\varepsilon\right), \\ Q\left(0,t,\varepsilon\right) = Q^0\left(t,\varepsilon\right). \end{cases}$$

Expand the functions $P^0(t,\varepsilon)$ and $Q^0(t,\varepsilon)$ is series of $\varepsilon-$:

$$\begin{cases} P^0(t,\varepsilon) = \sum\limits_{k=0}^{-\infty} P_k(t)\varepsilon^k, \\ Q^0(t,\varepsilon) = \sum\limits_{k=0}^{-\infty} Q_k(t)\varepsilon^k. \end{cases} \tag{3}$$

As in the system (3) we write the expansion of the system of equations by ε:

$$\begin{cases} P_i(z,t,\varepsilon) = \sum\limits_{k=0}^{-\infty} P_k(z,t)\varepsilon^k, \\ Q_i(z,t,\varepsilon) = \sum\limits_{k=0}^{-\infty} Q_k(z,t)\varepsilon^k. \end{cases} \tag{4}$$

And now take the system (4) into account in the system of equations (2) :

$$\begin{cases} \sum\limits_{k=0}^{-\infty} \frac{\partial P_{i,k}(z,t)}{\partial t}\varepsilon^k + \frac{c_i^2}{F_i}\sum\limits_{k=0}^{-\infty} \frac{\partial Q_{i,k}(z,t)}{\partial z}\varepsilon^{k+1} = 0, \\ \sum\limits_{k=0}^{-\infty} \frac{\partial Q_{i,k}(z,t)}{\partial t}\varepsilon^k + F_i\sum\limits_{k=0}^{-\infty} \frac{\partial P_{i,k}(z,t)}{\partial z}\varepsilon^{k+1} + 2a_i\sum\limits_{k=0}^{-\infty} Q_{i,k}(z,t)\,\varepsilon^k = 0. \end{cases} \tag{5}$$

Write the expressions of the degrees of the coefficient ε of the obtained system (5) in descending order [10], [12]:

$$\begin{cases} \frac{\partial P_{i,0}(z,t)}{\partial t}\varepsilon^0 + \frac{\partial P_{i,-1}(z,t)}{\partial t}\varepsilon^{-1} + \frac{\partial P_{i,-2}(z,t)}{\partial t}\varepsilon^{-2} + \frac{\partial P_{i,-3}(z,t)}{\partial t}\varepsilon^{-3} + ...+ \\ +\frac{c^2}{F_i}\left(\frac{\partial Q_{i,0}(z,t)}{\partial z}\varepsilon + \frac{\partial Q_{i,-1}(z,t)}{\partial z}\varepsilon^0 + \frac{\partial Q_{i,-2}(z,t)}{\partial z}\varepsilon^{-1} + \frac{\partial Q_{i,-3}(z,t)}{\partial z}\varepsilon^{-2} + ...\right) = 0, \\ \frac{\partial Q_{i,0}(z,t)}{\partial t}\varepsilon^0 + \frac{\partial Q_{i,-1}(z,t)}{\partial t}\varepsilon^{-1} + \frac{\partial Q_{i,-2}(z,t)}{\partial t}\varepsilon^{-2} + \frac{\partial Q_{i,-3}(z,t)}{\partial t}\varepsilon^{-3} + ...+ \\ +F_i\left(\frac{\partial P_{i,0}(z,t)}{\partial z}\varepsilon + \frac{\partial P_{i,-1}(z,t)}{\partial z}\varepsilon^0 + \frac{\partial P_{i,-2}(z,t)}{\partial z}\varepsilon^{-1} + \frac{\partial P_{i,-3}(z,t)}{\partial z}\varepsilon^{-2} + ...\right) + \\ 2a\left(Q_{i,0}(z,t)\varepsilon^0 + Q_{i,-1}(z,t)\varepsilon^{-1} + Q_{i,-2}(z,t)\varepsilon^{-2} + Q_{i,-3}(z,t)\varepsilon^{-3} + ...\right) = 0. \end{cases} \tag{6}$$

Writing appropriate solution to the coefficient ε^1-we obtain:

$$\begin{cases} \dfrac{\partial Q_{i,0}(z,t)}{\partial z} = 0, \\ \dfrac{\partial P_{i,0}(z,t)}{\partial z} = 0, \end{cases} \Rightarrow \begin{cases} Q_{i,0}(z,t) = \tilde{Q}_{i,0}(t), \\ P_{i,0}(z,t) = \tilde{P}_{i,0}(t). \end{cases} \tag{7}$$

Writing appropriate solution to the coefficient ε^0-we obtain:

$$\begin{cases} \dfrac{\partial P_{i,0}(z,t)}{\partial t} + \dfrac{c_i^2}{F_i}\dfrac{\partial Q_{i,-1}(z,t)}{\partial z} = 0, \\ \dfrac{\partial Q_{i,0}(z,t)}{\partial t} + F_i\dfrac{\partial P_{i,-1}(z,t)}{\partial z} + 2a_iQ_{i,0}(z,t) = 0, \end{cases}$$

here we determine the derivatives $\dfrac{\partial Q_{i,-1}(z,t)}{\partial z}$ and $\dfrac{\partial P_{i,-1}(z,t)}{\partial z}$ integrate with respect to z and get:

$$\begin{cases} \dfrac{\partial Q_{i,-1}(z,t)}{\partial z} = -\dfrac{F_i}{c_i^2}\tilde{P}'_{i,0}(t), \\ \dfrac{\partial P_{i,-1}(z,t)}{\partial z} = -\dfrac{1}{F_i}\tilde{Q}'_{i,0}(t) - \dfrac{2a_i}{F_i}\tilde{Q}_{i,0}(t), \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} Q_{i,-1}(z,t) = \tilde{Q}_{i,-1}(t) - \dfrac{F_i}{c_i^2}\tilde{P}'_{i,0}(t)z, \\ P_{i,-1}(z,t) = \tilde{P}_{i,-1}(t) - \left(\dfrac{1}{F_i}\tilde{Q}'_{i,0}(t) + \dfrac{2a_i}{F_i}\tilde{Q}_{i,0}(t)\right) z. \end{cases} \tag{8}$$

Writing the term corresponding to the coefficient ε^{-1}—we obtain:

$$\begin{cases} \dfrac{\partial Q_{i,-2}(z,t)}{\partial z} = -\dfrac{F_i}{c_i^2}\dfrac{\partial P_{i,-1}(z,t)}{\partial t}, \\ \dfrac{\partial P_{i,-2}(z,t)}{\partial z} = -\dfrac{1}{F_i}\dfrac{\partial Q_{i,-1}(z,t)}{\partial t} - \dfrac{2a_i}{F_i}Q_{i,-1}(z,t). \end{cases} \tag{9}$$

Determining the system (8), i.e. $\dfrac{\partial Q_{i,-1}(z,t)}{\partial t}$ and $\dfrac{\partial P_{i,-1}(z,t)}{\partial t}$ from (8) and take then into account in (9):

$$\begin{cases} \dfrac{\partial Q_{i,-2}(z,t)}{\partial z} = -\dfrac{F_i}{c_i^2}\tilde{P}'_{i,-1}(t) + \dfrac{1}{c_i^2}\tilde{Q}''_{i,0}(t)\,z + \dfrac{2a_i}{c_i^2}\tilde{Q}'_{i,0}(t)\,z, \\ \dfrac{\partial P_{i,-2}(z,t)}{\partial z} = -\dfrac{1}{F_i}\tilde{Q}'_{i,-1}(t) + \dfrac{1}{c_i^2}\tilde{P}''_{i,0}(t)\,z - \dfrac{2a_i}{F_i}\left(\tilde{Q}_{i,-1}(t) - \dfrac{F_i}{c_i^2}\tilde{P}'_{i,0}(t)\,z\right). \end{cases} \tag{10}$$

Integrating the obtained expression(10) with respect to z—we obtain:

$$\begin{cases} Q_{i,-2}(z,t) = -\dfrac{F_i}{c_i^2}\tilde{P}'_{i,-1}(t)\,z + \dfrac{1}{c_i^2}\tilde{Q}''_{i,0}(t)\dfrac{z^2}{2} + \dfrac{2a_i}{c_i^2}\tilde{Q}'_{i,0}(t)\dfrac{z^2}{2} + \tilde{Q}_{i,-2}(t), \\ P_{i,-2}(z,t) = -\dfrac{1}{F_i}\tilde{Q}'_{i,-1}(t)\,z + \dfrac{1}{c_i^2}\tilde{P}''_{i,0}(t)\dfrac{z^2}{2} + \tilde{P}_{i,-2}(t) - \dfrac{2a_i}{F_i}\tilde{Q}_{i,-1}(t)\,z + \dfrac{2a_i}{c_i^2}\tilde{P}'_{i,0}(t)\dfrac{z^2}{2}. \end{cases} \tag{11}$$

Grouping the terms of ε^{-2}—according to the above rule by equating their coefficient zero, we obtain:

$$\begin{cases} \dfrac{\partial P_{i,-2}(z,t)}{\partial t} + \dfrac{c_i^2}{F_i}\dfrac{\partial Q_{i,-3}(z,t)}{\partial z} = 0, \\ \dfrac{\partial Q_{i,-2}(z,t)}{\partial t} + F_i\dfrac{\partial P_{i,-3}(z,t)}{\partial z} + 2a_iQ_{i,-2}(z,t) = 0. \end{cases} \tag{12}$$

Now determine the derivative of the expression (11) with respect to t:

$$\begin{cases} \dfrac{\partial P_{i,-2}(z,t)}{\partial t} = -\dfrac{1}{F_i}\tilde{Q}''_{i,-1}(t)z + \dfrac{1}{c_i^2}\tilde{P}'''_{i,0}(t)\dfrac{z^2}{2} + \tilde{P}'_{i,-2}(t) - \dfrac{2a_i}{F_i}\tilde{Q}'_{i,-1}(t)z + \dfrac{2a_i}{c_i^2}\tilde{P}''_{i,0}(t)\dfrac{z^2}{2}, \\ \dfrac{\partial Q_{i,-2}(z,t)}{\partial t} = -\dfrac{F_i}{c_i^2}\tilde{P}''_{i,-1}(t)z + \dfrac{1}{c_i^2}\tilde{Q}'''_{i,0}(t)\dfrac{z^2}{2} + \dfrac{2a_i}{c_i^2}\tilde{Q}''_{i,0}(t)\dfrac{z^2}{2} + \tilde{Q}'_{i,-2}(t). \end{cases} \tag{13}$$

Take expression (13) into account in (12):

$$\begin{cases} \dfrac{\partial Q_{i,-3}(z,t)}{\partial z} = \dfrac{1}{c_i^2}\tilde{Q}''_{i,-1}(t)z - \dfrac{F_i}{c_i^4}\tilde{P}'''_{i,0}(t)\dfrac{z^2}{2} - \dfrac{F_i}{c_i^2}\tilde{P}'_{i,-2}(t) + \dfrac{2a_i}{c_i^2}\tilde{Q}'_{i,-1}(t)z - \dfrac{2a_iF_i}{c_i^4}\tilde{P}''_{i,0}(t)\dfrac{z^2}{2}, \\ \dfrac{\partial P_{i,-3}(z,t)}{\partial z} = \dfrac{1}{c^2}\tilde{P}''_{i,-1}(t)\,z - \dfrac{1}{F_ic_i^2}\tilde{Q}'''_{i,0}(t)\dfrac{z^2}{2} - \dfrac{2a_i}{F_ic_i^2}\tilde{Q}''_{i,0}(t)\dfrac{z^2}{2} - \dfrac{1}{F_i}\tilde{Q}'_{i,-2}(t) + \\ + \dfrac{2a_i}{c_i^2}\tilde{P}'_{i,-1}(t)z - \dfrac{2a_i}{c_i^2F_i}\tilde{Q}''_{i,0}(t)\dfrac{z^2}{2} - \dfrac{4a_i^2}{F_ic_i^2}\tilde{Q}'_{i,0}(t)\dfrac{z^2}{2} - \dfrac{2a_i}{F_i}\tilde{Q}_{i,-2}(t). \end{cases} \tag{14}$$

Finally, integrating the expression (14) with respect to the variable$z$we obtain:

$$\begin{cases} Q_{i,-3}(z,t) = \dfrac{1}{c_i^2}\tilde{Q}''_{i,-1}(t)\dfrac{z^2}{2} - \dfrac{F_i}{c_i^4}\tilde{P}'''_{i,0}(t)\,\dfrac{z^3}{6} - \dfrac{F_i}{c_i^2}\tilde{P}'_{i,-2}(t)\,z + \dfrac{2a}{c_i^2}\tilde{Q}'_{i,-1}(t)\,\dfrac{z^2}{2} - \\ \qquad\qquad - \dfrac{2a_iF_i}{c_i^4}\tilde{P}''_{i,0}\,(t)\dfrac{z_i^3}{6} + \tilde{Q}_{i,-3}(t), \\ P_{i,-3}(z,t) = \dfrac{1}{c_i^2}\tilde{P}''_{i,-1}\,(t)\,\dfrac{z^2}{2} - \dfrac{1}{F_ic_i^2}\tilde{Q}'''_{i,0}\,(t)\,\dfrac{z^3}{6} - \dfrac{2a_i}{F_ic_i^2}\tilde{Q}''_{i,0}\,(t)\dfrac{z^3}{6} - \dfrac{1}{F_i}\tilde{Q}'_{i,-2}(t)z + \\ + \dfrac{2a_i}{c_i^2}\tilde{P}'_{i,-1}(t)\,\dfrac{z_i^2}{2} - \dfrac{2a_i}{c_i^2F_i}\tilde{Q}''_{i,0}\,(t)\dfrac{z^3}{6} - \dfrac{4a_i^2}{F_ic_i^2}\tilde{Q}'_{i,0}(t)\dfrac{z^3}{6} - \dfrac{2a_i}{F_i}\tilde{Q}_{i,-2}(t)z + \tilde{P}_{i,-3}(t). \end{cases}$$

In this way, similarly we can determine $P_{i,-k}\,(z,t)$ and $Q_{i,-k}\,(z,t)$.

Now we give an algorithm for constructing the recurrent formula determining the parameters $P_{i,-k}(z,t)$ and $Q_{i,-k}(z,t)$ for any term of the series. Writing the expressions obtained from the coefficients of ε^{-k+1} in the system (5) by means of the matrices in the following from:

$$\frac{\partial}{\partial z}\begin{pmatrix} Q_{i,-k} \\ P_{i,-k} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{F_i}{c_i^2} \\ -\dfrac{1}{F_i} & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} Q_{i,-k+1} \\ P_{i,-k+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\dfrac{2a_i}{F_i} & 0 \end{pmatrix}\begin{pmatrix} Q_{i,-k+1} \\ P_{i,-k+1} \end{pmatrix}.$$

Here making the substatituons as $\begin{pmatrix} Q_{i,-k} \\ P_{i,-k} \end{pmatrix} = W_{-k}$, $\begin{pmatrix} 0 & -\dfrac{F_i}{c_i^2} \\ -\dfrac{1}{F_i} & 0 \end{pmatrix} = A$ and $\begin{pmatrix} 0 & 0 \\ -\dfrac{2a_i}{F_i} & 0 \end{pmatrix} = B$ and integrating the obtained expressions with respect to z- we get:

$$W_{-k}(z,t) = \tilde{W}_{-k}(t) + A\int_0^z \frac{\partial W_{-k+1}(\xi,t)}{\partial t}d\xi + B\int_0^z W_{-k+1}(\xi,t)d\xi. \qquad (15)$$

Hence by substituting k-with (k-1) and finding $W_{-k+1}(z,t)$- and taking it into account in this equation we can write the equation (15) in the following form and obtain:

$$W_{-k}(z,t) = \chi_1 + A^2\frac{\partial^2}{\partial t^2}\int_0^z W_{-k+2}(\eta,t)d\eta\int_\eta^z d\xi +$$

$$+ (AB+BA)\,\frac{\partial}{\partial t}\int_0^z W_{-k+2}(\eta,t)d\eta\int_\eta^z d\xi + B^2\int_0^z W_{-k+2}(\eta,t)d\eta\int_\eta^z d\xi. \qquad (16)$$

Here taking into account

$$\chi_1 = \tilde{W}_{-k}(t) + A\tilde{W}'_{-k+1}(t)\,z + B\tilde{W}_{-k+1}(t)\,z,$$

and changing the order of the integral in the expression (16) for the expression, (15) we obtain the following formula corresponding to the first iteration.

$$W_{-k}(z,t) = \chi_1 + A^2\frac{\partial^2}{\partial t^2}\int_0^z (z-\eta)\,W_{-k+2}(\eta,t)d\eta +$$

$$+ (AB + BA) \frac{\partial}{\partial t} \int_0^z (z - \eta)\ W_{-k+2}(\eta, t) d\eta + B^2 \int_0^z (z - \eta)\ W_{-k+2}(\eta, t) d\eta. \qquad (17)$$

Now, for determining $W_{-k+2}(z, t)$ corresponding to the second iteration, in the expression (15) instead of k-take $-k + 2$ and obtain:

$$W_{-k+2}(\eta, t) = \tilde{W}_{-k+2}(t) + A \int_0^\eta \frac{\partial W_{-k+3}(\xi, t)}{\partial t} d\xi + B \int_0^\eta W_{-k+3}(\xi, t) d\xi.$$

Taking this obtained expression into account in (17) and making some simplifications, we get:

$$W_{-k}(z, t) = \chi_1 + A^2 \tilde{W}''_{-k+2}(t)(-1)\ \frac{(z - \eta)^2}{2!}\bigg|_{\eta=0}^{z} +$$

$$+A^3 \frac{\partial^3}{\partial t^3} \int_0^z W_{-k+3}(\xi, t) d\xi \int_\xi^z (z - \eta)\, d\eta + A^2 B \frac{\partial^2}{\partial t^2} \int_0^z W_{-k+3}(\xi, t) d\xi \int_\xi^z (z - \eta)\, d\eta +$$

$$+ (AB + BA)\, \tilde{W}'_{-k+2}(t)(-1)\ \frac{(z - \eta)^2}{2!}\bigg|_{\eta=0}^{z} +$$

$$+ (AB + BA)\, A \frac{\partial^2}{\partial t^2} \int_0^z W_{-k+3}(\xi, t) d\,\xi \int_\xi^z (z - \eta)\, d\eta +$$

$$+ (AB + BA)\ B\, \frac{\partial}{\partial t} \int_0^z W_{-k+3}(\xi, t) d\,\xi \int_\xi^z (z - \eta)\, d\eta +$$

$$+B^2 \tilde{W}_{-k+2}(t)(-1)\ \frac{(z - \eta)^2}{2!}\bigg|_{\eta=0}^{z} + B^2 A \frac{\partial}{\partial t} \int_0^z W_{-k+3}(\xi, t) d\,\xi \int_\xi^z (z - \eta)\, d\eta +$$

$$+B^3 \int_0^z W_{-k+3}(\xi, t) d\,\xi \int_\xi^z (z - \eta)\, d\eta.$$

We get this expression by calculating the interior integral as in expression (16):

$$W_{-k}(z, t) = \chi_1 + A^2 \frac{z^2}{2} \tilde{W}''_{-k+2}(t) + (AB + BA)\, \frac{z^2}{2} \tilde{W}'_{-k+2}(t) + B^2 \frac{z^2}{2} \tilde{W}_{-k+2}(t) +$$

$$+A^3 \int_0^z \frac{(z - \xi)^2}{2!} \frac{\partial^3 W_{-k+3}(\xi, t)}{\partial t^3} d\xi + A^2 B \int_0^z \frac{(z - \xi)^2}{2!} \frac{\partial^2 W_{-k+3}(\xi, t)}{\partial t^2} d\xi +$$

$$+ (AB + BA)\, A \int_0^z \frac{(z - \xi)^2}{2!} \frac{\partial^2 W_{-k+3}(\xi, t)}{\partial t^2} d\xi +$$

$$+ (AB + BA)\, B \int_0^z \frac{(z - \xi)^2}{2!} \frac{\partial W_{-k+3}(\xi, t)}{\partial t} d\xi +$$

$$+B^2 A \int_0^z \frac{(z - \xi)^2}{2!} \frac{\partial W_{-k+3}(\xi, t)}{\partial t} d\xi + B^3 \int_0^z \frac{(z - \xi)^2}{2!} W_{-k+3}(\xi, t) d\xi. \qquad (18)$$

After making some groupping, we can write the expression (18) as follows:

$$W_{-k}(z,t)=\chi_1+\left[A^2\tilde{W''}_{-k+2}(t)+(AB+BA)\,\tilde{W'}_{-k+2}(t)+B^2\tilde{W}_{-k+2}(t)\right]\frac{z^2}{2}+$$

$$+A^2\sum_{n=0}^{1}A^nB^{1-n}\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^{2+n}W_{-k+3}(\xi,t)}{\partial t^{2+n}}d\xi+$$

$$+(AB+BA)\sum_{n=0}^{1}A^nB^{1-n}\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^{1+n}W_{-k+3}(\xi,t)}{\partial t^{1+n}}d\xi+$$

$$+B^2\sum_{n=0}^{1}A^nB^{1-n}\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^{n}W_{-k+3}(\xi,t)}{\partial t^{n}}d\xi=\chi_1+\frac{z^2}{2}\left(A\frac{\partial}{\partial t}+B\right)^2\tilde{W}_{-k+2}(t)+$$

$$+\int_0^z\frac{(z-\xi)^2}{2!}\left(A\frac{\partial}{\partial t}+B\right)^2\sum_{n=0}^{1}\left(A\frac{\partial}{\partial t}\right)^nB^{1-n}W_{-k+3}(\xi,t)d\xi.$$

Containing the calculating in the expression (15) we substitute $-k-$with $-k+3-$ and writing in the expression (18) we obtain:

$$W_{-k}(z,t)=\chi_1+\chi_2+$$

$$+A^3\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^3}{\partial t^3}\left[\tilde{W}_{-k+3}(t)+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi+$$

$$+A^2B\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^2}{\partial t^2}\left[\tilde{W}_{-k+3}(t)+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi+$$

$$+(AB+BA)\,A\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial^2}{\partial t^2}\left[\tilde{W}_{-k+3}(t)+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+\right.$$

$$\left.+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi+(AB+BA)\,B\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial}{\partial t}\left[\tilde{W}_{-k+3}(t)+\right.$$

$$\left.+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi+$$

$$+B^2A\int_0^z\frac{(z-\xi)^2}{2!}\frac{\partial}{\partial t}\left[\tilde{W}_{-k+3}(t)+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi+$$

$$+B^3\int_0^z\frac{(z-\xi)^2}{2!}\left[\tilde{W}_{-k+3}(t)+A\int_0^\xi\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta+B\int_0^\xi W_{-k+4}(\eta,t)d\eta\right]d\xi,\quad(19)$$

here $\chi_2 = \frac{z^2}{2!}\left(A\frac{\partial}{\partial t}+B\right)^2\tilde{W}_{-k+2}(t)$. Changing the order of the integral in the obtained expression (19) and making simplifications we get:

$$W_{-k}(z,t) = \chi_1 + \chi_2 + A^3\tilde{W}'''_{-k+3}(t)\frac{z^3}{3!} + A^4\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^4 W_{-k+4}(\eta,t)}{\partial t^4}d\eta +$$
$$+A^3B\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^3 W_{-k+4}(\eta,t)}{\partial t^3}d\eta + A^2B\tilde{W}'''_{-k+3}(t)\frac{z^3}{3!}+$$
$$+A^2BA\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^3 W_{-k+4}(\eta,t)}{\partial t^3}d\eta+$$
$$+A^2B^2\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^2 W_{-k+4}(\eta,t)}{\partial t^2}d\eta + (AB+BA)A\tilde{W}''_{-k+3}(t)\frac{z^3}{3!}+$$
$$+(AB+BA)A^2\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^3 W_{-k+4}(\eta,t)}{\partial t^3}d\eta+$$
$$+(AB+BA)AB\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^2 W_{-k+4}(\eta,t)}{\partial t^2}d\eta+$$
$$+(AB+BA)B\tilde{W}'_{-k+3}(t)\frac{z^3}{3!} + (AB+BA)BA\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^2 W_{-k+4}(\eta,t)}{\partial t^2}d\eta+$$
$$+(AB+BA)B^2\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta + B^2A\tilde{W}'_{-k+3}(t)\frac{z^3}{3!}+$$
$$+B^2A^2\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial^2 W_{-k+4}(\eta,t)}{\partial t^2}d\eta+$$
$$+B^2AB\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta + B^3\tilde{W}'_{-k+3}(t)\frac{z^3}{3!}+$$
$$+B^3A\int_0^z \frac{(z-\eta)^3}{3!}\frac{\partial W_{-k+4}(\eta,t)}{\partial t}d\eta + B^4\int_0^z \frac{(z-\eta)^3}{3!}W_{-k+4}(\eta,t)\,d\eta. \tag{20}$$

Making some groupping in the expression (20) we can write it in the following form:

$$W_{-k}(z,t) = \chi_1 + \chi_2 + \left[\left(A\frac{\partial}{\partial t}+B\right)^3\tilde{W}_{-k+3}(t)\right]\frac{z^3}{3!}+$$
$$+\left[\int_0^z \frac{(z-\eta)^3}{3!}\left(A\frac{\partial}{\partial t}+B\right)^3\sum_{n=0}^{1}\left(A\frac{\partial}{\partial t}\right)^n B^{1-n}\tilde{W}_{-k+4}(\eta,t)\right]d\eta. \tag{21}$$

Note that by this rule we can determine any step of the iteration. Based on the obtained expressions (21), (19) and (16) we can write the general recurrent formula as follows:

$$W_{-k}(z,t) = \int_0^z \frac{(z-\xi)^k}{k!}\left(A\frac{\partial}{\partial t}+B\right)^k\sum_{n=0}^{1}\left(A\frac{\partial}{\partial t}\right)^n B^{1-n}W_0(\xi,t)\,d\xi+$$

$$+\sum_{m=1}^{k}\frac{z^m}{m!}\left(A\frac{\partial}{\partial t}+B\right)^m\tilde{W}_{-k+m}(t)+\tilde{W}_{-k}(t)=\sum_{m=0}^{k}\frac{z^m}{m!}\left(A\frac{\partial}{\partial t}+B\right)^m\tilde{W}_{-k+m}(t)+$$

$$+\int_0^z\frac{(z-\xi)^k}{k!}\left(A\frac{\partial}{\partial t}+B\right)^k\sum_{n=0}^{1}\left(A\frac{\partial}{\partial t}\right)^n B^{1-n}W_0\left(\xi,t\right)d\xi.$$

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