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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE STURM-LIOUVILLE EQUATION WITH AN OSCILLATING POTENTIAL

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Abstract. *In the article we consider the Sturm-Liouville equation* $-y'' + (q(x) + h(x))y = 0$, $0 \le x < +\infty$, *where* $q(x)$ *satisfies the conditions of regularity of growth at infinity and* $h(x)$ *is a fast-oscillating function. Sufficient conditions are found under which the asymptotics of solutions of the equation is determined only by the function* $q(x)$. *A new method for obtaining asymptotic formulas for solutions is proposed, consisting in the standard replacement of the equation by a system of first-order equations followed by the application of the Hausdorff identity [1].*

Keywords: Sturm–Liouville equation, asymptotics of solutions, oscillating potential, Hausdorff identity

Mathematics Subject Classification (2020): 34B24, 34D05, 34E10

1. Introduction

The paper proposes a new approach to the study of the asymptotic behavior of solutions of the Sturm-Liouville differential equation

$$-y'' + (q(x) + h(x))y = 0. \tag{1}$$

It is well known if the function $q(x)$ has regular growth at infinity: $q'(x)$ and $q''(x)$ do not change sign for sufficiently large x and $q(x) \to +\infty$, $q'(x) = o(q^{\alpha}(x))$, $0 < \alpha < \frac{3}{2}$

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for $x \to +\infty$, then the equation $-y'' + q(x)y = 0$ has two linearly independent solutions for which the asymptotic WKB (Wentzel-Kramers-Brillouin) formulas are valid:

$$y_{1,2}(x) \sim \frac{1}{\sqrt[4]{q(x)}} e^{\pm i \int\limits_0^x \sqrt{q(t)}dt}, \quad x \to +\infty. \tag{2}$$

The main purpose of our note is to expand the class of Sturm-Liouville equations with an oscillating potential, for solutions of which one can write our asymptotic formulas for $x \to +\infty$.

2. Main Results

Let us consider the equation (1). We turn to a system of linear first order differential equations by introducing the vector column $Y = (y, y')$. Then equation (1) is equivalent to the following system of first-order differential equations:

$$Y' = (A_0 + A_1)Y, \tag{3}$$

where

$$A_0(x) = \begin{pmatrix} 0 & 1 \\ q(x) & 0 \end{pmatrix}, \quad A_1(x) = \begin{pmatrix} 0 & 0 \\ h(x) & 0 \end{pmatrix}.$$

In order to construct the asymptotics for the fundamental system of solutions of equation (3), we reduce it to almost diagonal form. The eigenvalues of the matrix $A_0(x)$ denote as $\lambda_1(x)$ and $\lambda_2(x)$. Then $\lambda_1(x) = \mu(x)$, $\lambda_2(x) = -\mu(x)$, where $\mu(x) := q^{\frac{1}{2}}(x)$.

Let

$$T(x) = \begin{pmatrix} \frac{1}{\mu} & -\frac{1}{\mu} \\ 1 & 1 \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} 1 & -1 \\ \mu & \mu \end{pmatrix}.$$

Then matrix $T(x)$ reduces the matrix $A_0(x)$ to the diagonale form

$$T^{-1} A_0 T = \Lambda(x),$$

where

$$\Lambda(x) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} = \mu \Lambda_0, \quad \Lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will make a replacement, assuming

$$Y = T \cdot Z, \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Then the system (3) will be rewritten as

$$Z'(x) = \Lambda + \frac{h(x)}{2\sqrt{q(x)}} G_0 - \frac{q'(x)}{4q(x)} F_0 Z(x),$$

where

$$\frac{h(x)}{2\sqrt{q(x)}} G_0 = T^{-1} A_1 T, \quad \frac{q'(x)}{4q(x)} F_0 = T^{-1} T',$$

$$G_0 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad F_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We pass to new variables, replacing both the independent variable x and the unknown vector function $Z(x)$:

$$\xi = \int_0^1 q^{\frac{1}{2}}(t)dt, \quad x = \varphi(\xi), \quad Z(x) = U(\xi).$$

Then if denote by

$$a(\xi) = \frac{h(x)}{2\sqrt{q(x)}}, \quad b(\xi) = \frac{q'(x)}{4q(x)},$$

we get the system

$$U'(\xi) = (\Lambda_0 + a(\xi)G_0 - b(\xi)F_0)\, U(x).$$

On function $a(\xi)$ we impose the condition

$$\int\limits_{\xi}^{\infty} a(t)dt < \infty, \quad \forall \xi > 0, \tag{4}$$

and introduce the function into consideration

$$a_1(\xi) = \int_\xi^\infty a(t)dt.$$

At the same time, improper integrals are understood in the sense of conditional convergence. Let us make another substitution using the matrix exponent

$$U(\xi) = e^{-a_1(\xi)G_0} \cdot V(\xi).$$

We get the system

$$V'(\xi) = \left(e^{a_1(\xi)G_0}\Lambda_0 e^{-a_1(\xi)G_0} - b(\xi)e^{a_1(\xi)G_0}F_0 e^{-a_1(\xi)G_0}\right) V(\xi). \tag{5}$$

Next we use the Hausdorff formula

$$e^{\xi A}Be^{-\xi A} = B + \xi[A,B] + \frac{\xi^2}{2!}[A,[A,B]] + \dots .$$

Since $G_0^2 = 0$, then endless rows break off, and we come to the system

$$V'(\xi) = (\lambda_0 - a_1(\xi)G_{11} - b(\xi)F_0 + b(\xi)a_1(\xi)F_{11})\, V(\xi), \tag{6}$$

where

$$G_{11} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad F_{11} = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} = -2G_0.$$

We will require the following conditions to be met:

$$\int_\xi^\infty a_1(t)dt < \infty, \quad \xi > 0, \tag{7}$$

$$b(\xi)a_1(\xi) \in L_1(0,\infty). \tag{8}$$

We rewrite the matrix $\Lambda_0 - b(\xi)F_0$ in the form

$$\Lambda_0 - b(\xi)F_0 = \tilde{\Lambda}_0 - b(\xi)\tilde{F}_0,$$

$$\tilde{\Lambda}_0 = \begin{pmatrix} 1+\frac{q'}{4q} & 0 \\ 0 & -1+\frac{q'}{4q} \end{pmatrix}, \quad \tilde{F}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = F_0 - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The system (6) will take the form

$$V'(\xi) = \left(\tilde{\Lambda}_0 - b(\xi)\tilde{F}_0 + b(\xi)a_1(\xi)F_{11}\right)V(\xi),$$

where $F_2(\xi) = -a_1(\xi)G_{11} + b(\xi)a_1(\xi)F_{11}$.

We introduce into consideration the matrix T_1, for which it is true:

$$V = (I+T_1)P, \quad T_1\tilde{\Lambda}_0 = \tilde{\Lambda}_0 T_1 - b(\xi)\tilde{F}_0,$$

$$T_1(\xi) = b(\xi)\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

We arrive at the system

$$P' = \left(\tilde{\Lambda}_0 - (I+T_1)^{-1}T_1' + (I+T_1)^{-1}F_2(I+T_1)\right)P. \tag{9}$$

By conditions (4), (7), (8) system (9) has L-diagonale form, the coefficient matrix on the right side differs from the diagonal matrix Λ_0 by the summable matrix. Then we can apply Levinson theorem to the system (9). Therefore, the fundamental system of solutions of a system of differential equations (9) for $x \to +\infty$ has the following asymptotics:

$$P(\xi) = \exp\int_0^{\xi}\tilde{\lambda}_0(t)dt \cdot (I + C + o(1)),$$

where matrix C has the form

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, we obtain the following theorem.

Theorem. *Suppose that $q(x)$ satisfies the conditions of regularity of growth at infinity and the conditions (4), (7) and (8) are met. Then for the fundamental system of solutions of equation (1) for $x \to +\infty$ asymptotic formulas (2) are valid.*

Example 1. Let $q(x) = x^{\alpha}$, $\alpha > 0$, $h(x) = x^{\alpha}\sin x^{\beta}$. Then simple calculations using integration by parts show that conditions (4), (7) and (8) are satisfied if $\beta > \frac{\alpha}{2} + 1$.

Example 2. Let $q(x) = x^{\alpha}$, $\alpha > 0$, $h(x) = x^{\alpha}\sin e^{x}$. When conditions (4), (7) and (8) are met.

Therefore, in both examples, $h(x)$ does not affect the asymptotics of solutions of equation (1).

Remark 1. Previously, similar studies were conducted by other methods in [2] and [3]. In [2], the asymptotics of solutions were sought using the Liouville transform. At the same time, more stringent conditions were imposed on the function $h(x)$ compared to ours. In [3], the study of the Sturm-Liouville equation was reduced to the study of the Riccati equation. The result obtained in [3] for Example 1 repeats our result, but in the general situation our results are more general.

Remark 2. It is interesting to note that if the potential $q(x)$ is an irregular function, then our method allows in some cases to replace $q(x)$ with the average function of M.Otelbaev or with the Steklov averaging.

Remark 3. Note that our method can be applied without significant changes in the case when $q(x) \to -\infty$ at $x \to +\infty$ and satisfies the conditions of regularity of growth.

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