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ABOUT ONE BOUNDARY VALUE PROBLEM FOR A NON-CLASSIC TYPE DIFFERENTIAL EQUATION

M.E. KERIMOVA, M.M. SABZALIYEV*

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Abstract. *A boundary value problem stated in an infinite semi-strip for a third order non-classic type differential equation degenerated into a hyperbolic equation, is considered. The complete expansion of the solution of the problem with respect to a small parameter as constructed and the residual term is estimated.*

Keywords: boundary value problem, non-classic type differential equation, hyperbolic equation, small parameter, boundary layer type function, residual term

Mathematics Subject Classification (2020): 35B25, 35C20

1. Introduction

When passing from one physical characteristics to another one, mathematical models of some nonsmoth processes one described by differential equations with a small parameter in front of higher order derivatives. Such equations one said to be singularly perturbed differential equations. First studies in this field belong to acad. A. N. Tikhonov. He has studied the criterion of convergence of the solution of a boundary value problem for a system of singularly perturbed ordinary differential equations to the solution of the degenerated problem obtained when a small parameter approaches to a zero. After Tikhonov's this research interest to theory of singularly perturbed equations has grown even more. In this field we can show the studies of L. S. Pontryagin, O. A. Oleinik, O. A. Ladyzhenskaya, K. O. Friedrichs and others. But a great majority of studies were devoted to classic type singularly perturbed differential equations. But there is a little research in the field of non-classic type singularly perturbed differential equations. In this paper,

* Corresponding author.

Mahbuba E. Kerimova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: maxbuba3773@mail.ru

Mahir M. Sabzaliyev
Baku Business University, Baku, Azerbaijan;
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: sabzalievm@mail.ru

we consider a boundary value problem stated for a non-classic type singularly perturbed quasilinear differential equation on an infinite semistrip.

2. Main Results

On the infinite semistrip $P = \{ (x, y) | \, 0 \le x \le 1, 0 \le y < +\infty \}$ we consider the following boundary value problem:

$$L_\varepsilon u \equiv \varepsilon^2 \frac{\partial}{\partial x}(\Delta u) - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + F(x, y, u) = 0, \tag{1}$$

$$u|_{x=0} = u|_{x=1} = 0, \left. \frac{\partial u}{\partial x} \right|_{x=1} = 0 \quad (0 \le y < +\infty), \tag{2}$$

$$u|_{y=0} = 0, \lim_{y \to +\infty} u = 0 \quad (0 \le x \le 1). \tag{3}$$

Here $\varepsilon > 0$ is a small parameter, $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplace operator, $F(x, y, u)$ is a given function. The goal of the paper is to construct the asymptotic of the solution of the problem (1)-(3) with respect to a small parameter.

To construct the asymptotics of the stated problem, we carry out iterative processes. In the first iterative process the approximate solution of the equation (1) is sought in the form

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + ... \varepsilon^n W_n = \sum_{i=0}^{n} \varepsilon^i W_i.$$

In order to determine the functions W_i, $i = 1, 2, ..., n$ we use the expansion (1) of the operator L_ε and obtain the following recurrent equations:

$$\frac{\partial W_0}{\partial x} + \frac{\partial W_0}{\partial y} + F(x, y, W_0) = 0, \tag{4}$$

$$\frac{\partial W_i}{\partial x} + \frac{\partial W_i}{\partial y} + \frac{\partial F(x, y, W_0)}{\partial W_0} W_i = h_i, \quad i = 1, 2, ..., n. \tag{5}$$

For solving differential equations (4), (5) we will use the following boundary conditions:

$$W_i|_{x=0} = 0, W_i|_{y=0} = 0, \quad i = 0, 1, 2, ..., n.$$

If the function $F(x, y, u)$ is rather smooth in the domain $(x, y) \in P \times (-\infty, +\infty)$ and satisfies the conditions

$$\frac{\partial F(x, y, u)}{\partial u} \ge \gamma^2 u, \tag{6}$$

$$F(x, y, u)|_{x=y} = 0, \left. \frac{\partial^k F(x, y, 0)}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}} \right|_{x=y} = 0, \tag{7}$$

then the boundary value problem (4) has a unique solution and this solution is a smooth function in the domain P. When in (7) $k = k_1 + k_2 + k_3,\ k = 0, 1, ..., 2(n+1)$, $W_0(x, y) \in C^{2(n+1)}(P)$, and the function $W_0(x, y)$ satisfies the condition

$$\left.\frac{\partial^k W_0(x,y)}{\partial x^{k_1} \partial y^{k_2}}\right|_{y=x} = 0, \quad k = k_1 + k_2, \quad k = 0, 1, ..., 2(n+1). \tag{8}$$

If the function $F(x, y, u)$ in addition to the condition (7) satisfies the condition

$$\left|\frac{\partial^k F(x,y,u)}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}}\right| \leq c_1 e^{-\lambda y}, \quad c_1 = const > 0, \quad \lambda = const > 0, \tag{9}$$

as well, the function $W_0(x, y)$ being the solution of the boundary value problem (4) in addition to the condition (8) satisfies the condition

$$\left|\frac{\partial^k W_0(x,y)}{\partial x^{k_1} \partial y^{k_2}}\right| \leq c_2 e^{-\lambda y}, \quad c_2 = const > 0, \quad k = k_1 + k_2 = 0, 1, ..., 2(n+1), \tag{10}$$

as well.

For $k = 0$ from condition (10) we obtain that for the function $W_0(x, y)$ the condition $\lim\limits_{y \to +\infty} W_0(x, y) = 0$ is satisfied.

The boundary value problems (5) where the functions $W_i, i = 1, 2, ...n$, are determined, are linear. The condition (8) satisfied by the function $W_0(x, y)$ provides the solvability of boundary value problems (5) and ensures that functions $W_i(x, y)$ satisfy the conditions

$$\left.\frac{\partial^k W_i(x,y,)}{\partial x^{k_1} \partial y^{k_2}}\right|_{x=y} = 0, \quad i = 1, 2, ..., n. \tag{11}$$

Furthermore, according to the condition (10) satisfied by the function $W_0(x, y)$ it is proved that the functions $W_i(x, y)$ satisfy the conditions.

According to this condition, we can say that the functions $W_i(x, y)$, $i = 1, 2, ...n$, also satisfy the condition $\lim\limits_{y \to +\infty} W_i(x, y) = 0$, $i = 1, 2, ...n$.

Thus, in the first iterative process, we construct such a function that this function satisfies the boundary conditions

$$W|_{x=0} = 0, \quad W|_{y=0} = 0, \quad \lim_{y \to +\infty} W = 0.$$

But the function need not satisfy the boundary conditions in (2) on $x = 1$. To satisfy the boundary conditions lost on $x = 1$, we construct a boundary layer type function. The boundary layer function is constructed in the second iterative process. For that making the substitution of the variables $1 - x = \varepsilon$, $y = 7$ we write a new expansion of the operator $L_{\varepsilon,1}$ in the coordinates (t, y). A boundary layer type function near $x = 1$ is sought as an approximate solution of the equation

$$L_{\varepsilon,1}(W + V) - L_{\varepsilon,1} W = o(\varepsilon^{n+1})$$

in the form $V = \sum_{j=0}^{n+1} \varepsilon^j V_j$. The functions $V_0, V_1, ..., V_{n+1}$ contained in this expansion are determined as boundary layer type solutions of the following boundary value problems:

$$\frac{\partial^3 V_0}{\partial t^3} + \frac{\partial^2 V_0}{\partial t^2} + \frac{\partial V_0}{\partial t} = 0, \quad V_0|_{t=0} = -W_0(1,y), \quad \left.\frac{\partial V_0}{\partial t}\right|_{t=0} = 0,$$

$$\frac{\partial^3 V_j}{\partial t^3} + \frac{\partial^2 V_j}{\partial t^2} + \frac{\partial V_j}{\partial t} = g_j(t,y), \quad V_j|_{t=0} = \varphi_j(y), \quad \left.\frac{\partial V_j}{\partial t}\right|_{t=0} = \psi_j(y).$$

Here $g_j(t,y)$ is a known function dependent on the functions $V_0, V_1, ..., V_{n+1}$. Since the number of the roots of characteristic equations with negative real part corresponding to the ordinary differential equations equals the number of the boundary conditions lost in the first iterative process, we can say that the boundary problem (1)-(3) regularly degenerates on the boundary $x = 1$.

The function V determined in such a way ensures satisfaction of the conditions $(W+V)|_{x=1} = 0$ and $\frac{\partial}{\partial x}(W+V)\big|_{x=1} = 0$.

Multiply all the functions V_j, $j = 0, 1, ..., n+1$, by smoothing functions and retain the previous denotations for the obtained new functions. As the expense of smoothing functions all the functions V_j, $j = 0, 1, ..., n+1$, become zero for $x = 0$.

Based on the conditions (10), (11) it is proved that the constructed functions V_j satisfy the conditions $\lim_{y\to+\infty} V_j = 0$, $j = 0, 1, ..., n$, as well. According to the formula where the functions V_j, $j = 0, 1, ..., n+1$, are determined, if the conditions

$$\left|\frac{\partial^k W_i(1,0)}{\partial x^{k_1} \partial y^{k_2}}\right| = 0, \quad k = k_1 + k_2, \quad k + i \le n+1, \quad i = 0, 1, ..., n, \tag{12}$$

are satisfied, then $V_j|_{y=0} = 0$, $j = 0, 1, ..., n+1$. If the function $F(x,y,u)$ satisfies the condition

$$\left|\frac{\partial^k F(1,0,0)}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}}\right| = 0, \quad k = k_1 + k_2 + k_3, \tag{13}$$

then the condition (12) is also satisfied. Thus, the constucted sum $\tilde{u} = W + V$ satisfies the boundary conditions

$$(W+V)|_{x=0} = (W+V)|_{x=1} = 0, \quad \frac{\partial}{\partial x}(W+V)|_{x=1} = 0,$$

$$(W+V)|_{y=0} = 0, \quad \lim_{y\to+\infty}(W+V) = 0.$$

Denoting the difference of the exact solution u of the problem (1)-(3) and the constructed approximate solution $\tilde{u}$ by $u - \tilde{u} = z$, we get the following asymptotic expansion of the solution of the considered boundary value problem:

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + z. \tag{14}$$

It can be easily shown that the function satisfies all boundary conditions in the considered boundary value problem. Acting on the determination formula of the function z by the operator L_ε , for this function we can obtain the following estimation:

$$\varepsilon^2 \int_0^{+\infty} \left(\frac{\partial z}{\partial x}\Big|_{x=0} \right)^2 dy + \varepsilon \iint_p \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dxdy + c_1 \iint_p z^2 dxdy \le c_2.$$

Here the constants $c_1 > 0, c_2 > 0$ are independent of ε . Thus, the results obtained in this paper are expressed in the form of the following theorem.

Theorem 1. *Assume that $F(x, y, u) \in C^{2(n+2)}(P \times (0; +\infty))$, and conditions (6),(7),(9) and (13) are satisfied. Then the asymptotic expansion (14) is valid for the solution of the boundary value problem (1)-(3). Here the functions W_i are determined in the first iterative process, the functions V_j are boundary layer type functions near $x = 1$ and are determined in the second iterative process, z is a residual.*

The linear case of the stated problem was considered and the complete asymptotic of the solution of the problem was constructed in [2]. In the paper [1], constructing an inner boundary layer type function near the bisectrix of the first quarter of the solution of a boundary value problem stated in a rectangular domain for the equation (1), the first terms of the asymptotics of the considered boundary value problem were constructed. In [3] the complete asymptotics of the solution of a boundary value problem stated in a rectangular domain was constructed for the equation (1).

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