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TRANSVERSE SHEAR OF AN ISOTROPIC ELASTIC MEDIUM IN THE CASE WHEN THE BINDER AND INCLUSIONS ARE WEAKENED BY CRACK INITIATION

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Abstract. *An elastic medium is considered, weakened by a doubly periodic system of round holes, filled with absolutely rigid inclusions, soldered along the bypass and has a crack initiation. The medium (binder) is weakened by two periodic systems of rectilinear crack initiation directed collinear to the abscissa and ordinate axes, and their sizes are not the same. General representations are constructed that describe a class of problems with a doubly periodic stress distribution outside circular holes and cracks under transverse shear. The analysis of the limiting equilibrium of cracks in the framework of the end zone model is carried out on the basis of a nonlocal fracture criterion with a force condition for the propagation of the crack tip and a deformation condition for determining the advancement of the edge of the end zone of the crack.*

Basic resolving equations are obtained in the form of infinite algebraic systems and three nonlinear singular integro-differential equations.

The equations in each approximation were solved by the Gaussian method with the choice of the principal element for different values of the order of M, depending on the radius of the holes. Calculations were carried out to determine the forces in the connections of the end zones and the ultimate loads causing the growth of cracks.

Keywords: doubly periodic lattice, average stresses, boundary conditions, transverse shear, linear algebraic equations, singular equations, crack initiation

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1. Introduction

Currently, in many branches of modern technology, technical means are used in the form of perforated elements. In this regard, the development of methods for calculating the strength of perforated elements of machines and structures with cracks is of great importance. The study of these issues is important in connection with the development of energy, the chemical industry and other branches of technology, as well as the widespread use of materials with a periodic structure (composites).

A model of crack initiation in composites with a doubly periodic structure, based on the consideration of the zone of the crack formation process, is proposed.

It is believed that the zone of the fracturing process is a layer of finite length containing material with partially broken bonds between individual structural elements. The presence of bonds between the shores of the pre-fracture zone (the zone of weakened inter-partial bonds of the material) is modeled by the application of adhesion forces caused by the presence of bonds to the surface of the pre-fracture zone. The analysis of the limiting equilibrium of the prefracture zone during transverse shear is performed on the basis of the criterion of the limiting shear of the material bonds and includes:

1) establishing the dependence of the adhesion forces on the shear of the prefracture zone banks;
2) assessment of the stress state near the pre-fracture zone, taking into account external loads and adhesion forces, as well as the location of rigid inclusions;
3) determination of the dependence of critical external loads on the geometric parameters of the composite medium at which a crack appears.

2. Formulation of the Problem

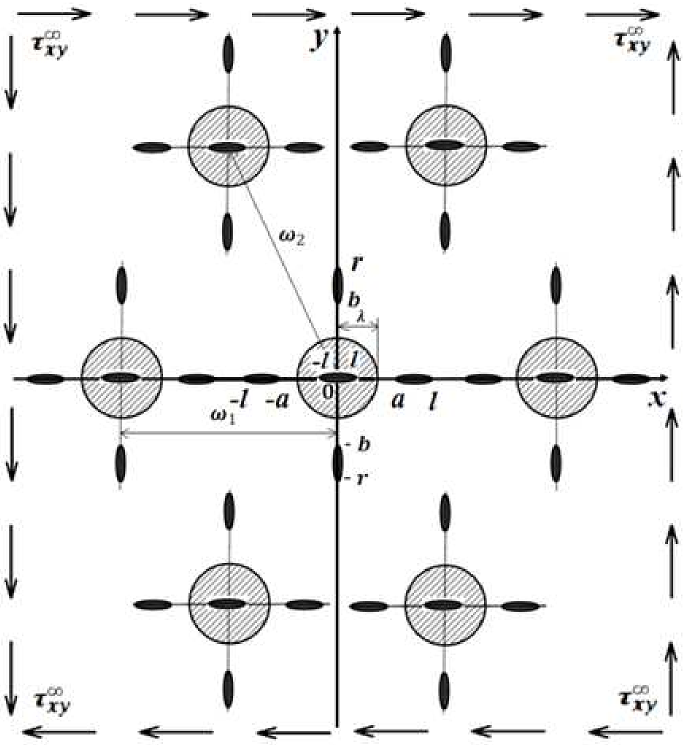
An elastic plane D is considered, weakened by a doubly periodic system of circular holes with radii λ ($\lambda < 1$) and the centers of these holes are at the points

$$P_{mn} = m\omega_1 + n\omega_2 \quad (m, n = 0, \pm 1, \pm 2, ...),$$

$$\omega_1 = 2, \quad \omega_2 = \omega_1 h e^{i\alpha}, \quad h > 0, \quad Im\omega_2 > 0.$$

Elastic washers made of a different material are soldered into the circular holes without interference. The plane under consideration is subjected to transverse shear by forces D (Fig. 1). As the external load increases in such a plane, zones of increased stresses are formed around the holes, the arrangement of which has a double periodic character. Cracks can occur in areas of increased stress. The problem of crack initiation is an important problem in damage mechanics.

The statement of this problem significantly expands the original concept of A. Griffiths, according to which a material always contains a large number of the smallest cracks. The formation (nucleation) of a crack under load corresponds to the data of fractographic observations. As the intensity of the external load increases, pre-fracture zones appear near the holes, which are modeled by areas with weakened interparticle bonds in the material. The interaction of the shores of these zones is modeled by introducing bonds between the shores of the pre-fracture zone with a given deformation diagram. The physical nature of such bonds and the size of the pre-fracture zones depend on the type of material. Since the indicated zones (interlayers of overstressed material) are small compared to the rest of the isotropic medium, weakened by a doubly periodic system of circular holes, they can be mentally removed by replacing them with cuts, the surfaces of which interact with each other according to a certain law corresponding to the action of the removed material.



Fiq. 1. Calculation scheme for the problem of crack initiation

Taking these effects into account in problems of fracture mechanics is an important but very difficult problem [1].

In the case under study, the emergence of an incipient crack in a medium weakened by a doubly periodic system of circular holes is a process of transition of the pre-fracture region into the region of broken bonds between the surfaces of the material. In this case, the size of the pre-destruction zone is not known in advance and must be determined.

Studies [2], [3], [11], [20]-[22] of the appearance of regions with a damaged material structure show that in the initial stage, the pre – fracture zones are a narrow – elongated layer, and then, with an increase in the load, a secondary system of zones containing material with partially broken bonds.

For a mathematical description of crack initiation in an isotropic medium weakened by a doubly periodic system of circular holes filled with rigid inclusions in the case under consideration, we come to the problem of the theory of elasticity for a medium when the medium contains pre–fracture zones. The pre–fracture zones are oriented in the direction of the maximum shear stresses. It is believed that in an isotropic medium there are two periodic systems of rectilinear pre–fracture zones collinear to the abscissa and ordinate axes (Fig. 1) of unequal length. The interaction of the banks of the pre–fracture zone (bonds between the banks) inhibits crack initiation. For a mathematical description of the interaction of the banks of the pre-fracture zone, it is assumed that there are connections between them, the deformation law of which is given.

Modeling the tip regions of cracks consists in considering them as part of the cracks and explicitly applying to the surfaces of the cracks in the tip zones of the adhesion forces that restrain their shear. The sizes of the end zones of the cracks are considered commensurate in comparison with the length of the cracks. The interaction of the edges of the end zones of cracks is modeled by introducing bonds with a given deformation diagram between the edges of the pre-fracture zone. The physical nature of such bonds and the size of the pre-fracture zones depend on the type of material. Under the action of an external load on a composite body in the bonds connecting the banks of the pre-failure zones, tangential forces $q_x(x)$, $q_y(y)$ in the plane and $q_x^0(x)$ in the inclusion, respectively, arise. These stresses are unknown in advance and must be determined from the solution of the boundary value problem of fracture mechanics according to the boundary conditions expressing the absence of elastic displacements along the bypass of the circular holes and the conditions on the banks of the pre-fracture zones, respectively [5].

The boundary conditions in the problem under consideration have the form

$$\begin{aligned} (\sigma_r - i\tau_{r\theta})_{b|\Omega_m} &= (\sigma_r - i\tau_{r\theta})\, s_{b|\Omega_m}, \\ (u + iv)_{b|\Omega_m} &= (u + iv)_{s|\Omega_m}, \end{aligned} \tag{1}$$

on the banks of cracks with end zones

$$\begin{aligned} (\sigma_y - i\tau_{xy})_s &= f_x(x) \quad \text{collinear abscissa}, \\ (\sigma_x - i\tau_{xy})_s &= f_y(y) \quad \text{collinear} \;\; y-\text{axis}, \\ (\sigma_x - i\tau_{xy})_b &= f_x^0(x) \quad \text{at}, \quad y = 0, \quad |x| \le \ell. \end{aligned} \tag{2}$$

Here Ω_m is the inclusion-plane interface in the cell with the number m; the values relating to the inclusion (washer) and the plane are hereinafter designated by the subscripts b and s; $f_x(x) = 0$ on free banks of cracks; $f_x(x) = -iq_x(x)$ on the banks of the end zones of cracks collinear to the abscissa axis; $f_y(y)$ on the free shores of cracks collinear with the ordinate $f_y(y) = -iq_y(y)$ on the shores of the end zones of the cracks collinear with the ordinate; $f_x^0(x) = 0$ on the free banks of the crack in the inclusion; $f_x^0(x) = -iq_x^0(x)$ on the banks of the end zones of cracks in the inclusion [10].

The main relations of the problem posed must be supplemented with relations connecting the displacement of the banks of the zones before the destruction and the tangential forces in the bonds.

Without loss of generality, we represent these relations in the form

$$u_s^+(x,\ 0) - u_s^-(x,\ 0) = C\,(x,\ q_x(x))\,q_x(x)$$

for end zones of cracks collinear to the abscissa axis, (3)

$$\upsilon_s^+(0,\ y) - \upsilon_s^-(0,\ y) = C\,(y,\ q_y(y))\,q_y(y)$$

for end zones of cracks collinear with the ordinate axis,

$$u_b^+(x,\ 0) - u_b^-(x,\ 0) = C_0\left(x,\ q_x^0(x)\right)q_x^0(x) \tag{4}$$

for end zones of cracks in the inclusion,

where functions $C\,(x, q_x(x))$ and $C\,(y, q_y(y))$ represent effective bond compliance; $(u_s^+ - u_s^-)$ is the shift of the banks of the pre-fracture zones of the collinear abscissa axis; $(\upsilon_s^+ - \upsilon_s^-)$ is the shift of the banks of the pre-fracture zones of the collinear $y-$axis; $\left(u_b^+ - u_b^-\right)$ is the displacement of the edges of the end zones of the crack in the inclusion.

To determine the limiting value of the external load at which the crack nucleation occurs, the problem statement must be supplemented with the condition (criterion) for the appearance of a crack (rupture of inter-partial bonds in the material). As such a condition, we take the criterion of the critical shear of the banks of the pre-fracture zone

$$\begin{aligned} u_s^+ - u_s^- &= \delta_{IIc} \ \text{ on } \ L_1, \\ \upsilon_s^+ - \upsilon_s^- &= \delta_{IIc} \ \text{ on } \ L_2, \\ \upsilon_b^+ - \upsilon_b^- &= \delta_{IIc} \ \text{ at } \ y = 0, \quad |x| \le \ell, \end{aligned} \tag{5}$$

where δ_{IIc} is the characteristic of the resistance of the medium material to cracking; L_1 is a set of pre–fracture zones collinear to the abscissa axis; L_2 is a set of pre–fracture zones collinear to the ordinate axis; L_3 is set of pre-destruction zones, including [6], [7].

3. Method for Solving the Problem

To solve the problem, the method developed in solving the doubly periodic elastic problem [4] is naturally combined with the method [21] for constructing in explicit form the Kolosov–Muskhelishvili potentials corresponding to unknown tangential displacements along the pre-fracture zones.

Stresses and displacements in the flat theory of elasticity can be represented [18], [20] through two analytical functions of a complex variable $z = x + iy$, $\Phi(z)$ and $\Psi(z)$ using the Kolosov–Muskhelishvili formulas:

$$\begin{aligned} \sigma_y + \sigma_x = \sigma_r + \sigma_\theta &= 2\left[\Phi\,(z) + \overline{\Phi\,(z)}\right], \\ \sigma_y - \sigma_x + 2i\tau_{xy} = e^{-2i\theta}\,(\sigma_\theta - \sigma_r + 2i\tau_{r\theta}) &= 2\left[\bar{z}\Phi'\,(z) + \Psi\,(z)\right], \\ 2\mu\,(u + i\upsilon) &=: \varphi\,(z) - z\overline{\Phi\,(z)} - \overline{\psi\,(z)}, \\ \varphi'\,(z) = \Phi\,(z)\,, &\quad \psi'\,(z) = \Psi\,(z)\,, \end{aligned} \tag{6}$$

where μ is the shear modulus of the material; ν is Poisson's coefficient; $k = 3 - 4v$ for plane deformation; $k = (3 - v) / (1 + v)$ for a plane stress state; r, θ is polar coordinates.

We represent the stresses and displacements in terms of the Kolosov–Muskhelishvili potentials $\Phi(z)$ and $\Psi(z)$ (6). Based on formulas (6) and boundary conditions on the contours of circular holes (1) and pre-fracture zones (2), the problem is reduced to determining two analytic functions $\Phi(z)$ and $\Psi(z)$ in the domain D from the boundary conditions (t, t_1 and t_0 are the affixes of the bank points of the pre-fracture zones collinear to the abscissa and ordinate axes, respectively) [17]

$$\Phi(\tau) + \overline{\Phi(\tau)} - \left[\bar{\tau}\Phi'(\tau) + \Psi(\tau)\right] e^{2i\theta} = \Phi_0(\tau) + \overline{\Phi_0(\tau)} - \left[\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)\right] e^{2i\theta}, \tag{7}$$

$$-k_s\overline{\Phi(\tau)} + \Phi(\tau) - \left[\bar{\tau}\Phi'(\tau) + \Psi(\tau)\right] e^{2i\theta} = \frac{\mu_s}{\mu_b}\left\{-k_b\overline{\Phi_0(\tau)} + \Phi_0(\tau) - \left[\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)\right] e^{2i\theta}\right\}, \tag{8}$$

$$\Phi_0(t_0) + \overline{\Phi_0(t_0)} + t_0\overline{\Phi_0'(t_0)} + \overline{\Psi_0(t_0)} = f_x^0(t_0), \tag{9}$$

$$\begin{gathered} \Phi(t) + \overline{\Phi(t)} + t\overline{\Phi'(t)} + \overline{\Psi(t)} = f_x(t), \\ \Phi(t_1) + \overline{\Phi(t_1)} + t_1\overline{\Phi'(t_1)} + \overline{\Psi(t_1)} = f_y(t_1), \end{gathered} \tag{10}$$

where $\tau = \lambda e^{i\theta} + m\omega_1 + n\omega_2$, $m, n = 0 \pm 1, \pm 2, ...$, μ_s, k_s and μ_b, k_b are shear moduli and Muskhelishvili constant for the plane and inclusion, respectively [9].

4. Solution of the Boundary Value Problem

The solution to the boundary value problem (7)–(10) is sought in the form [12]

$$\begin{gathered} \Phi_0(z) = \Phi_{01}(z) + \Phi_{02}(z), \quad \Psi_0(z) = \Psi_{01}(z) + \Psi_{02}(z), \\ \Phi_{01}(z) = \tfrac{1}{2\pi i}\int_{-\ell}^{\ell} \tfrac{g_0(t_0)}{t_0 - z} dt_0, \quad \Phi_{02}(z) = i\sum_{k=0}^{\infty} a_{2k} z^{2k}, \end{gathered} \tag{11}$$

$$\Psi_{01}(z) = \frac{1}{2\pi i}\int_{-\ell}^{\ell}\left[\frac{g_0(t_0)}{t_0 - z} - \frac{t_0 g_0(t_0)}{(t_0 - z)^2}\right] dt_0, \quad \Psi_{02}(z) = i\sum_{k=0}^{\infty} b_{2k} z^{2k}, \tag{12}$$

$$\begin{gathered} \Phi(z) = \Phi_1(z) + \Phi_2(z) + \Phi_3(z), \\ \Psi(z) = \Psi_1(z) + \Psi_2(z) + \Psi_3(z), \end{gathered} \tag{13}$$

$$\begin{gathered} \Phi_1(z) = i\tau_{xy}^{\infty} + i\alpha_0 + i\sum_{k=0}^{\infty} \alpha_{2k+2} \tfrac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!}, \\ \Psi_1(z) = i\tau_{xy}^{\infty} + i\sum_{k=0}^{\infty} \beta_{2k+2} \tfrac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!} - i\sum_{k=0}^{\infty} \alpha_{2k+2} \tfrac{\lambda^{2k+2} S^{(2k+1)}(z)}{(2k+1)!}, \end{gathered} \tag{14}$$

$$\begin{gathered} \Phi_2(z) = \tfrac{1}{2\omega}\int_{L_1} g(t)\ ctg\tfrac{\pi}{\omega}(t - z)\ dt, \\ \Psi_2(z) = -\tfrac{\pi z}{2\omega^2}\int_{L_1} g(t)\sin^{-2}\tfrac{\pi}{\omega}(t - z)\ dt, \end{gathered} \tag{15}$$

$$\begin{gathered}\Phi_3(z) = -\tfrac{i}{2\omega}\int_{L_2} g_1(t_1)\ ctg\tfrac{\pi}{\omega}(it_1 - z)\, dt_1,\\ \Psi_3(z) = -\tfrac{i}{2\omega}\int_{L_2}\left\{g_1(t_1)\, ctg\tfrac{\pi}{\omega}(it_1 - z) + \right.\\ \left. + \left[ctg\,\frac{\pi}{\omega}(it_1 - z) + \frac{\pi}{\omega}(2t_1 + z)\sin^2\frac{\pi}{\omega}(it_1 - z)\right] g_1(t_1)\right\} dt_1,\end{gathered} \tag{16}$$

where

$$\rho(z) = \left(\frac{\pi}{\omega}\right)^2 \sin^{-2}\left(\frac{\pi}{\omega}z\right) - \frac{1}{3}\left(\frac{\pi}{\omega}\right)^2, \quad \sum_m{}' \left[\frac{P_m}{(z - P_m)^2} - \frac{2z}{P_m} - \frac{1}{P_m}\right],$$

the prime at the sum means that the index $m = 0$ is excluded during the summation; the integrals in (15), (16) are taken along the lines $L_1 = [-\ell, -a]\cup[a, \ell]$, $L_2 = [-r,\ -b]\cup[b, r]$, $g(t)$, $g_1(t_1)$ and $g_0(t_0)$ are the sought functions characterizing the displacement of the banks of the zones before the destruction

$$\begin{gathered} g(x) = -\tfrac{2\mu_s i}{1+k_s}\tfrac{d}{dx}\left[u_s^+(x,0) - u_s^-(x,0)\right],\\ g_1(y) = \tfrac{2\mu_s}{1+k_s}\tfrac{d}{dy}\left[\upsilon_s^+(0,y) - \upsilon_s^-(0,y)\right],\end{gathered} \tag{17}$$

$$g_0(x) = -\frac{2\mu_b i}{1 + k_b}\frac{d}{dx}\left[u_b^+(x,0) - u_b^-(x,0)\right]. \tag{18}$$

Relations (11)–(18) should be supplemented with additional conditions arising from the physical meaning of the problem

$$\begin{gathered}\int_{-\ell}^{\ell} g_0(t_0)\, dt_0 = 0, \quad \int_{-h}^{-a} g(t)\, dt = 0, \quad \int_a^h g(t)\, dt = 0,\\ \int_{-r}^{-b} g_1(t_1)\, dt_1 = 0, \quad \int_b^r g_1(t_1)\, dt_1 = 0.\end{gathered} \tag{19}$$

Let us present the dependencies which the coefficients of expressions (11)–(16) must satisfy.

From the conditions of antisymmetry with respect to the coordinate axes, we find that

$$Im\alpha_{2k} = 0, \quad Im\beta_{2k} = 0 \quad (k = 1, 2, ...).$$

From the condition of the constancy of the principal vector of all forces acting on the arc connecting two congruent points in D, it follows [22]

$$\alpha_0 = \frac{\pi^2}{24}\beta_2\lambda^2.$$

It is easy to verify that functions (11)–(16) under condition (19) define a class of problems with a doubly periodic stress distribution. Unknown functions $g(x)$, $g_1(y)$ and $g_0(x)$ and constants α_{2k} and β_{2k} must be determined from the boundary conditions (7) and (10). Due to the fulfillment of the periodicity conditions, the system of boundary conditions (7) is replaced by one functional equation, for example, on the contour $\tau = \lambda e^{i\theta}$, and the system of boundary conditions (9), (10) is replaced by the boundary conditions on the contours L_1 and L_2.

To compose equations for the coefficients α_{2k}, β_{2k} of the functions $\Phi_1(z)$ and $\Psi_1(z)$, we represent the boundary condition (7), (8) in the form

$$\Phi_1(\tau) + \overline{\Phi_1(\tau)} - \left[\bar{\tau}\Phi_1'(\tau) + \Psi_1(\tau)\right] e^{2i\theta} = \sum_{k=-\infty}^{\infty} A_{2k} e^{2ki\theta} + \varphi_1(\theta) + f_1(\theta), \tag{20}$$

where

$$\begin{gathered} \varphi_1(\theta) = -\Phi_2(\tau) - \overline{\Phi_2(\tau)} + \left[\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)\right] e^{2i\theta}, \\ f_1(\theta) = -\Phi_3(\tau) - \overline{\Phi_3(\tau)} + \left[\bar{\tau}\Phi_3'(\tau) + \Psi_3(\tau)\right] e^{2i\theta}, \\ \Phi_0(\tau) + \overline{\Phi_0(\tau)} - \left[\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)\right] e^{2i\theta} = \sum_{k=-\infty}^{\infty} A_{2k} e^{2ki\theta}. \end{gathered} \tag{21}$$

With respect to the functions $\varphi_1(\theta)$ and $f_1(\theta)$, we will assume that they expand on $|\tau| = \lambda$ in Fourier series. Due to antisymmetry, these series have the form

$$\begin{gathered} \varphi_1(\theta) = \sum_{k=-\infty}^{\infty} C_{2k}' e^{2ki\theta}, ReC_{2k}' = 0, \\ f_1(\theta) = \textstyle\sum_{k=-\infty}^{\infty} C_{2k}'' e^{2ki\theta}, ReC_{2k}'' = 0, \\ C_{2k}' = \tfrac{1}{2\pi}\int_0^{2\pi} \varphi_1(\theta)\, e^{-2ki\theta} d\theta, k = 0,\ \pm1,\ \pm2, \ldots, \\ C_{2k}'' = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\, e^{-2ki\theta} d\theta. \end{gathered} \tag{22}$$

Substituting here relations (21) and changing the order of integration, after calculating the integrals using the theory of residues, we find

$$C_{2k}' = -\frac{1}{2\omega}\int_{L_1} g(t)\varphi_{2k}(t)dt, C_{2k}'' = -\frac{1}{2\omega}\int_{L_1} g_1(t_1) f_{2k}(t_1)dt_1, \tag{23}$$

where

$$\begin{gathered} \varphi_0(t) = (1+\varepsilon)\,\gamma(t), \varphi_2(t) = -\frac{\lambda^2}{2}\gamma^{(2)}(t), \\ \varphi_{2k}(t) = -\frac{(2k-1)\,\lambda^{2k}}{(2k)!}\gamma^{(2k)}(t) + \frac{\lambda^{2k-2}}{(2k-3)!}\gamma^{(2k-2)}(t), \\ \varphi_{-2k}(t) = \frac{\varepsilon\lambda^{2k}}{(2k)!}\gamma^{(2k)}(t), \quad k = 1, 2, ..., \\ \gamma(t) = \operatorname{ctg}\frac{\pi}{\omega}t, \quad f_0(it_1) = \left[\delta(it_1) - \overline{\delta(it_1)}\right]\frac{(1+\varepsilon)}{2}, \\ f_2(it_1) = -\frac{\lambda^2}{2}\delta^{(2)}(it_1) + 2\left[\delta(it_1) - it\delta'(it_1)\right], \end{gathered}$$

$$f_{2k}(it_1) = \frac{(1-2k)\,\lambda^{2k}}{(2k)!}\delta^{(2k)}(it_1) + \frac{2\lambda^{2k-2}}{(2k-2)!}\left[k\delta^{(2k-2)}(it_1) - it_1\delta^{(2k-1)}(it_1)\right], \ k = 2, 3, \ldots,$$

$$f_{-2k}(it_1) = -\frac{\varepsilon\lambda^{2k}}{(2k)!}\overline{\delta^{(2k)}(it_1)}, \delta(it_1) = \mathrm{ctg}\frac{\pi}{\omega}(it_1), \varepsilon = 1, \quad k = 1, 2, \dots .$$

Substituting in the left side of the boundary condition (20) instead of $\Phi_1(z)$, $\overline{\Phi_1(z)}$, $\Phi_1'(z)$ and $\Psi_1(z)$ their expansions in Laurent series in the vicinity of $z = 0$, and in the right side of (20) Fourier series (22) and comparing the coefficients at the same powers of $e^{i\theta}$, we obtain two infinite systems algebraic equations for the coefficients α_{2k} and β_{2k}

$$i\alpha_{2j+2} = \sum_{j=0}^{\infty} iA_{j,k}\alpha_{2k+2} + b_j, \quad j = 0, 1, 2, ..., \tag{24}$$

where the expression for $A_{j,k}$, b_j and the constitutive relations for the coefficients β_{2k} are given in (20) and (21). Moreover, with the difference that in the ratios defining b_j and β_{2k} instead of A'_{2k}, in this case, M'_{2k} should be taken

$$M'_0 = M_0 - 2i\tau_{xy}^{\infty}, \quad M'_2 = M_2 + i\tau_{xy}^{\infty},$$

$$M'_{2k} = M_{2k}, \quad k = 1, \pm 2, \pm 3, ...,$$

$$M_{2k} = A_{2k} + C_{2k}, \quad C_{2k} = C'_{2k} + C''_{2k}, k = 0, \pm 1, \pm 2, \dots .$$

Proceeding similarly with the boundary condition (1), after some transformations we obtain the same system of equations as (24) for α^*_{2j+2} at $\varepsilon = -k$, and instead of the coefficients A'_{2k}, in this case, A^*_{2k}:

$$A^*_0 = (k-1)\, i\tau_{xy}^{\infty} + \frac{(1-k_0)\,\mu}{2\mu_b}A_0 - \frac{(1+k_0)\,\mu}{2\mu_0}B_0,$$

$$A^*_2 = i\tau_{xy}^{\infty} + C^*_2 + \frac{\mu}{\mu_0}A_2, \quad A^*_{2k} = C^*_{2k} + \frac{\mu}{\mu_0}A_{2k}, \quad k = 2, 3, ...,$$

$$A^*_{-2k} = C^*_{-2k} - \frac{\mu k_0}{\mu_0}A_{-2k} + \frac{(1+k_0)\,\mu}{\mu_0}B_{-2k}, \quad k = 1, 2, \dots .$$

Here $C^*_{2k}(k = 0, \pm 1, \pm 2, ...)$ are determined from (23) for $\varepsilon = -k_s$.

Using the obtained relations and performing some transformations, we obtain formulas that determine the coefficients α_{2k}, β_{2k}, A_0, A_{-2k}, through the quantities A_{2k}, as well as an infinite system of linear algebraic equations with respect to A_{2k}:

$$i\alpha_{2j+2} = \frac{1-\mu_s/\mu_b}{1+k_s}A_{2j+2}, \tag{25}$$

$$A_{-2j} = \frac{\mu_b}{\mu_b + k_0\mu_s}\left(C^*_{-2j} - C_{-2j}\right) - \frac{(1+k_b)\,\mu_s}{\mu_b - \mu_s k_b}B_{-2j} +$$

$$+\frac{\mu_b - \mu_s}{\mu_b + k_b\mu_s}\sum_{k=0}^{\infty}\lambda^{2j+2k+2}r_{j,k}A_{2k+2}, \quad A_0 = \sum_{k=0}^{\infty}e_{0,k}\lambda^{2k+2}A_{2k+2} + ie_0 - e_1,$$

$$e_{0,k} = \frac{1-\mu_s/\mu_b}{(1-2K_2\lambda^2)\,e}r_{0,k}, \quad e_0 = \frac{1+k_s}{(1+2K_2\lambda^2)\,e},$$

$$e_1 = \frac{1}{e} C_0^* - \frac{1 + (k_s - 1) K_2 \lambda^2}{(1 - 2K_2\lambda^2)} C_0 - \frac{(1 + k_0)\, \mu_s}{2\mu_b e} B_0,$$

$$e = \frac{1 + k_s}{2(1 - 2K_2\lambda^2)} - \frac{k_s - 1}{2} + \frac{\mu_s (k_b - 1)}{2\mu_b},$$

$$i\beta_{2j+2} = \frac{1 - \mu_s/\mu_b}{1 + k_s} \left[(2j + 3) A_{2j+2} + \right.$$

$$\left. + \sum_{k=0}^{\infty} \frac{(2j + 2k + 3)!\, g_{j+k+2} \lambda^{2j+2k+4} A_{2k+2}}{(2j + 2)!\,(2k + 1)!\, 2^{2j+2k+4}} \right] - A_{-2j-2} - C_{-2j-2},$$

$$A_{2j+2} = \sum_{k=0}^{\infty} D_{j,k} A_{2k+2} + T_j, \quad j = 0, 1, 2, \ldots, \tag{26}$$

$$D_{j,k} = (2j + 1) \lambda^{2j+2k+2} S_{j,k}/\gamma,$$

$$S_{j,k} = \frac{1 - \mu_s/\mu_b}{1 + k_s} \left(\gamma_{j,k} + \frac{\mu_b}{k_b \mu_s} \gamma^*_{j,k} + d_{j,k} \right), \quad d_{j,k} = \frac{g_{j+1} g_{k+1}}{2^{2j+2k+4}} \lambda^2 \eta\left(\mu_s/\mu_b\right),$$

$$\eta\left(\mu_s/\mu_b\right) = \frac{\frac{k_b - 1}{k_b} \cdot \frac{1}{1 - (k_s - 1) K_2 \lambda^2} - \frac{2}{1 - 2K_2\lambda^2}}{1 - (1 - 2K_2\lambda^2) \left[\frac{k_s - 1}{k_s + 1} - \frac{\mu_s}{\mu_b} \frac{(k_b - 1)}{(k_s + 1)} \right]},$$

$$T_0^* = \left(1 - \frac{\mu_b}{k_0 \mu_s}\right) i\tau_{xy}^{\infty}, \quad T_j = \left(T_j^* + h_j + K_j\right)/\gamma,$$

$$h_0 = \frac{1 + k_b}{k_b} \sum_{k=0}^{\infty} \frac{g_{k+2} \lambda^{2k+4}}{2^{2k+4}} B_{-2k-2},$$

$$K_0 = C_2 + \frac{\mu_0}{k_b \mu_s} C_2^* - \sum_{k=0}^{\infty} \frac{g_{k+2} \lambda^{2k+4}}{2^{2k+4}} \left(C_{2k-2} + \frac{\mu_b}{k_b \mu_s} C^*_{-2k-2} \right),$$

$$T_j^* = \frac{(2j + 1)\, g_{j+1} \lambda^{2j+2}}{2^{2j+2}} \eta_1\left(\mu_s/\mu_b\right) i\tau_{xy}^{\infty},$$

$$\eta_1\left(\mu_s/\mu_b\right) = \frac{(1 + \mu_b/k_b\mu_s) \left[(\mu_b/\mu_s)(k_b - 1) - (k_s - 1) \right]}{1 + (k_s - 1) K_2 \lambda^2 + (\mu_s/2\mu_b)(k_b - 1)(1 - 2K_2\lambda^2)},$$

$$h_j = \frac{(2j + 1)\ g_{j+1} \lambda^{2j+2}}{2^{2j+2}} (1 + k_b) \left\{ \frac{1}{2k_b \left[1 + (k_s - 1) K_2 \lambda^2\right]} + \right.$$

$$\left. + \frac{\mu_s}{2e\mu_b} \left[\frac{1}{1 - 2K_2\lambda^2} + \frac{1 - k_b}{2k_b \left[1 + (k_s - 1) K_2 \lambda^2\right]} \right] \right\} B_0 +$$

$$+ \frac{1 + k_b}{k_b} \sum_{k=0}^{\infty} \frac{(2j + 2k + 3)!\, g_{j+k+2} \lambda^{2k+2j+4}}{(2j)!\,(2k + 3)!\, 2^{2j+2k+4}} B_{-2k-2},$$

$$K_j = \frac{(2j + 1)\, g_{j+1} \lambda^{2j+2}}{2^{2j+2}} \left\{ \frac{1 - (k_s - 1) K_2 \lambda^2}{e} \left[\frac{1}{1 - 2K_2\lambda^2} + \frac{1 - k_0}{2k_s \left[1 + (k_s - 1) K_2 \lambda^2\right]} \right] - \right.$$

$$-1\} C_0 - \frac{(2j+1)\, g_{j+1}\lambda^{2j+2}}{2^{2j+2}} \left\{ \frac{1}{(1-2K_2\lambda^2)\, e} + \frac{1}{k_b\left[1+(k_s-1)\,K_2\lambda^2\right]} \times \right.$$

$$\times \left(\frac{1-k_b}{2e} + \frac{\mu_s}{\mu_b} \right) \Bigg\} C_0^* + C_{2j+2} \frac{\mu_b}{k_b\mu_s} C_{2j+2}^* - \sum_{k=0}^{\infty} \frac{(2j+2k+3)\,! g_{j+k+2}\lambda^{2j+2k+4}}{(2j)\,!\,(2k+3)\,!\,2^{2j+2k+4}} \times$$

$$\times \left(C_{-2k-2} + \frac{\mu_b}{k_b\mu_s} C_{-2k-2}^* \right),$$

$$\gamma = \frac{(1-\mu_s/\mu_b)\,(1-k_s\mu_b/k_b\mu_s)}{1+k_s} - \frac{1+k_b}{k_b}.$$

Here B_{2k} are defined in (25), $C_{2k}-$ in (23) at $\varepsilon=1$, C_{2k}^*- in (23) at $\varepsilon=-k_s$, $\gamma_{j,k}$ and $\gamma_{j,k}^*$ are defined in (26) at $\varepsilon=1$ and $\varepsilon=-k_s$, respectively.

Now, substituting (13)–(16) into the boundary condition (9), (10) on the crack faces with end zones, after some transformations we obtain a system of three singular integral equations for the sought functions $g(x)$, $g_1(y)$ and $g_0(x)$:

$$\frac{1}{\omega} \int_{L_1} g\,(t)\;K\,(t-x)\;dt + H(x) = f_x(x),$$

$$K(t-x) = ctg\frac{\pi}{\omega}\,(t-x)\,,$$

$$H(x) = x\overline{\Phi_s'(x)} + \overline{\Psi_s'(x)}, \Phi_s(x) = \Phi_1(x) + \Phi_3(x), \Psi_s(x) = \Psi_1(x) + \Psi_3(x),$$

$$-\frac{\pi}{\omega^2} \int_{L_2} g_1\,(t)\,(t-y)\;sh^{-2}\frac{\pi}{\omega}\,(t-y)\,dt + N(y) = f_y(y), \tag{27}$$

$$N(y) = iy\overline{\Phi_*'\,(iy)} + \overline{\Psi_*\,(iy)} + \Phi_*\,(iy) + \overline{\Phi_*\,(iy)},$$

$$\Phi_*(z) = \Phi_1(z) + \Phi_2(z), \Psi_*(z) = \Psi_1(z) + \Psi_2(z),$$

$$\Phi_*(z) = \Phi_1(z) + \Phi_2(z), \Psi_*(z) = \Psi_1(z) + \Psi_2(z),$$

$$\frac{1}{\pi} \int_{-\ell}^{\ell} \frac{g(t)dt}{t-x} + M(x) = -f_0(x),$$

$$M(x) = x\overline{\Phi_{01}'(x)} + \overline{\Psi_{02}(x)}.$$

Systems (25) and (26), together with singular integral equations (27), are the main resolving equations of the problem, which make it possible to determine the functions $g(x)$, $g_1(y)$ and $g_0(x)$ and the coefficients α_{2k}, β_{2k}.

5. Numerical Solution Technique and Analysis

Using the expansion of the functions $\operatorname{ctg}\frac{\pi}{\omega}z$, $sh^{-2}\frac{\pi}{\omega}z$ in the main strip of periods, as well as using a change of variables, after some transformations, we reduce the singular integral equations to the standard form. Using quadrature formulas, we reduce the basic resolving equations (25)–(27) to a set of two infinite systems of linear algebraic equations and to two finite algebraic systems with respect to the approximate values $P_k^0 = g(\eta_k)$, $(k = 1, 2, ..., M)$, R_v^0, $v = 1, 2, ..., M$, S_v^0, $i = 1, 2, ..., M$ of the required functions at the nodal points [15], [16]:

$$\sum_{v=1}^{M} A_{m,k} P_k^0 - \frac{1}{2} H_*(\eta_m) = -iq_x(\eta_m), \ k = 1, 2, ..., M-1, \tag{28}$$

$$\sum_{v=1}^{M} B_{m,v} R_v^0 + \frac{1}{2} N_*(\eta_m) = -iq_y(\eta_m), \ k = 1, 2, ..., M-1, \tag{29}$$

$$\sum_{k=1}^{M} F_{mi} S_i^0 + \frac{1}{2} H(\eta_m) = -\frac{1}{2} q_0(\eta_m), \ k = 1, 2, ..., M-1, \tag{30}$$

here

$$A_{m,k} = \frac{1}{2M}\left[\frac{1}{\sin\theta_m}\operatorname{ctg}\frac{\theta_m + (-1)^{|m-v|}\theta_v}{2} + B(\eta_m, \tau_v)\right],$$

$$\theta_m = \frac{2m-1}{2M}\pi, \quad (m = 1, 2, ..., M), \quad \tau_m = \cos\theta_m, \quad \eta_m = \tau_m, \quad \lambda_1 = \frac{a}{\ell},$$

$$B(\eta, \tau) = -\frac{1-\lambda_1^2}{2}\sum_{j=0}^{\infty} g_{j+1}\left(\frac{\ell}{2}\right)^{2j+2} u_0^j A_j,$$

$$A_j = \left[(2j+1) + \frac{(2j+1)(2j)(2j-1)}{1\cdot 2\cdot 3}\left(\frac{u}{u_0}\right) + \ldots + \right.$$

$$\left. + \frac{(2j+1)(2j)(2j-1)\ldots[(2j+1)-(2j+1-1)]}{1\cdot 2\ldots(2j+1)}\left(\frac{u}{u_0}\right)^j\right],$$

$$u = \frac{1-\lambda_1^2}{2}(\tau+1) + \lambda_1^2, u_0 = \frac{1-\lambda_1^2}{2}(\eta+1) + \lambda_1^2,$$

$$B_{mv} = \frac{1}{2M}\left[\frac{1}{\sin\theta_m}\operatorname{ctg}\frac{\theta_m + (-1)^{|m-v|}\theta_v}{2} + B_*(\eta_m, \tau_v)\right],$$

$$B_*(\eta, \tau) = -\frac{1-\lambda_2^2}{2}\sum_{j=0}^{\infty}(-1)^j(2j+1)g_{j+1}\left(\frac{r}{2}\right)^{2j+2} u_1^j A_j',$$

$$A_j' = \left\{(2j+1) + \frac{(2j+1)(2j)(2j-1)}{1\cdot 2\cdot 3}\left(\frac{u_1}{u_2}\right) + \ldots + \left(\frac{u_1}{u_2}\right)^j\right\},$$

$$u_1 = \frac{1-\lambda_2^2}{2}(\tau+1)+\lambda_2^2, \quad u_2 = \frac{1-\lambda_2^2}{2}(\eta+1)+\lambda_2^2, \lambda_2 = \frac{B}{r},$$
$$F_{m,i} = \frac{1}{2M}\frac{1}{\sin\theta_m}\operatorname{ctg}\frac{\theta_m+(-1)^{|m-k|}\theta_k}{2}.$$

To the resulting system of equations (28)–(30) are added additional equations (19) written in discrete form

$$\sum_{k=1}^{n}\frac{p_k^o}{\sqrt{\frac{1}{2}\left(1-\lambda_1^2\right)(\tau_k+1)+\lambda_1^2}} = 0,$$
$$\sum_{v=1}^{n}\frac{R_v^o}{\sqrt{\frac{1}{2}\left(1-\lambda_2^2\right)(\tau_v+1)+\lambda_2^2}} = 0. \tag{31}$$

The right-hand sides of the obtained algebraic systems (28), (29), and (30) include unknown values of stresses $q_x(\eta_m)$, $q_y(\eta_m)$, and $q_x^0(\eta_m)$ at the nodal points belonging to the end zones before fracture. Unknown values of stresses in bonds arising on the banks of pre-failure zones are determined from additional conditions (3)–(4). Using the obtained solution, relations (3)–(4) can be written in the form

$$g_0(x) = -\frac{2\mu_b i}{1+k_b}\frac{d}{dx}\left[C\left(x, q_x^0(x)\right) q_x^0(x)\right],$$
$$g(x) = -\frac{2\mu_s i}{1+k_s}\frac{d}{dx}\left[C\left(x, q_x(x)\right) q_x(x)\right], \tag{32}$$
$$g_1(y) = \frac{2\mu_s}{1+k_s}\frac{d}{dy}\left[C\left(y, q_y(y)\right) q_y(y)\right].$$

These equations are used to determine the stresses in the bonds. We represent these equations (32) in the following form

$$\frac{1+k_0}{2\mu_0 i}\int_\ell^x g_0(x)dx = C\left(x,\ q_x^0(x)\right)\ q_x^0(x), \quad x\in(-\ell,\ell),$$
$$\frac{1+k}{2\mu i}\int_h^x g(x)dx = C\left(x,\ q_x(x)\right) q_x(x), \quad x\in(-l,\ -a)\cup(a,l), \tag{33}$$
$$-\frac{1+k}{2\mu}\int_r^y g_1(y)dy = C\left(y,\ q_y(y)\right) q_y(x), \quad y\in(-r,\ -b)\cup(b,r).$$

To construct the missing equations, we require the fulfillment of conditions (33) at the corresponding nodal points belonging to the end zones $(l_1,l)\cup(a,a_1)$, $(r,r_1)\cup(b,b_1)$. As a result, we obtain three algebraic systems for finding the approximate values of $q_x^0(\eta_{m_0})$ $(m_0 = 1,2,\ldots,M_0)$, $q_x(\eta_{m_1})$ $(m_1 = 1,2,\ldots,M_1)$ and $q_y(\eta_{m_2})$ $(m_2 = 1,2,\ldots,M_2)$, respectively [13], [14]:

$$\begin{cases} C_0^* p_0^0(\eta_1) = C\left(\eta_1,\ q_x^0(\eta_1)\right)\ q_x^0(\eta_1), \\ C_0^*\left(p_0^0(\eta_1)+p_0^0(\eta_2)\right) = C\left(\eta_2,\ q_x^0(\eta_2)\right)\ q_x^0(\eta_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ C_0^*\sum\limits_{m_0=1}^{M_0} p_0^0(\eta_{m_0}) = C\left(\eta_{M_0},\ q_x^0(\eta_{M_0})\right)\ q_x^0(\eta_{M_0}), \end{cases}$$

$$\begin{cases} C_1^* p_0(\eta_1) = C(\eta_1,\ q_x(\eta_1))\ q_x(\eta_1), \\ C_1^* (p_0(\eta_1) + p_0(\eta_2)) = C(\eta_2,\ q_x(\eta_2))\ q_x(\eta_2), \\ \dots\dots\dots \\ \dots\dots\dots \\ C_1^* \sum\limits_{m_1=1}^{M_1} p_0(\eta_{m_1}) = C(\eta_{M_1},\ q_x(\eta_{M_1}))\ q_x(\eta_{M_1}), \end{cases}$$

$$\begin{cases} C_2^* R_0(\eta_1) = C(\eta_1,\ q_y(\eta_1))\ q_y(\eta_1), \\ C_2^* (R_0(\eta_1) + p_0(\eta_2)) = C(\eta_2,\ q_y(\eta_2))\ q_y(\eta_2), \\ \dots\dots\dots \\ \dots\dots\dots \\ C_2^* \sum\limits_{m_2=1}^{M_2} R_0(\eta_{m_2}) = C(\eta_{M_2},\ q_y(\eta_{M_2}))\ q_y(\eta_{M_2}). \end{cases}$$

Here

$$C_0^* = (1 + k_b)\,\pi\,(\ell - \ell_1)\,/\,(2\mu_b M),$$
$$C_1^* = (1 + k_s)\,\pi\,(h - a)\,/\,(2\mu_s M),$$
$$C_2^* = -(1 + k_s)\,\pi\,(b - r)\,/\,(2\mu_s M).$$

For the closedness of the obtained algebraic equations, five equations are lacking, which determine the sizes of the end zones of the cracks. Since the solution of the integral equations is sought in the class of everywhere bounded functions (stresses), it is necessary to add the conditions for the boundedness of stresses at the ends of the end zones of cracks to the systems obtained. These conditions are of the form [19]:

$$\begin{aligned} &\sum_{k=1}^{M} (-1)^k\, p_{0k}^0 ctg\frac{\theta_k}{2} = 0, \\ &\sum_{k=1}^{M} (-1)^k\, p_k^0 ctg\tfrac{\theta_k}{2} = 0, \\ &\sum_{k=1}^{M} (-1)^{v+M}\, R_v^0 tg\tfrac{\theta_v}{2} = 0, \\ &\sum_{k=1}^{M} (-1)^v\, R_v^0 ctg\frac{\theta_c}{2} = 0. \end{aligned} \tag{34}$$

The resulting systems of equations (25), (26), (28)–(31) completely determine the solution of the problem. For the numerical implementation of the described method, calculations were performed. Each of the infinite systems has been truncated to five equations. In the numerical calculations, $M = 30$ was assumed, which corresponds to the division of the integration interval into 30 Chebyshev nodes. Since the sizes of the pre-fracture zones are unknown, the solving algebraic system of equations (25), (26), (28)–(31) of the problem is nonlinear even with linear constraints. To solve it, the method of successive approximations is used, the essence of which is as follows: we solve the combined algebraic system for some definite values of the sizes of the pre-destruction zones with respect to the remaining unknowns. The rest of the unknowns enter the resolving system in a linear manner. The accepted values of the dimensions of the pre-fracture zones and the corresponding values of the remaining unknowns will, generally speaking, not satisfy the conditions for the bounded stresses at the tops of the pre-fracture zones. Therefore, choosing the values of the sizes of the zones before destruction, we will repeat the calculations many times until the conditions of bounded stresses (34) are satisfied with a given accuracy. In the case of a nonlinear law of bond deformation, an iterative algorithm similar to the method of elastic solutions [11] was used to determine the tangential forces in the pre-fracture zones.

To determine the limiting state at which the growth of cracks occurs, the condition of critical shear of the crack faces is used. Using the solution obtained, the conditions that determine the ultimate external load are the following:

$$\begin{aligned} &C\left(\lambda^0, q_x^0(\lambda^0)\right) q_x^0(\lambda) = \delta_c^0, \\ &C\left(\lambda_*, q_y(\lambda_*)\right) q_x(\lambda_*) = \delta_c, \\ &C\left(\lambda_*^1, q_x(\lambda_*^1)\right) q_y(\lambda_*^1) = \delta_c. \end{aligned} \qquad (35)$$

where δ_c^0, and δ_c- are the characteristics of the crack resistance of the inclusion material and the binder, respectively; λ^0, λ_* and λ_*^1 are the coordinates of the points at the base of the pre-fracture zones for the inclusion and the binder, respectively.

The analysis of the limiting equilibrium state of a piecewise homogeneous medium, at which a crack grows, is reduced to a parametric study of the combined algebraic system and the criterion for crack growth (35) for various laws of bond deformation, elastic constants of materials, and geometric characteristics of a perforated body.

It is believed that the law of deformation of interparticle bonds in the pre-fracture zone is linear at $(u^+ - u^-) \leq u_*$ and $(\upsilon^+ - \upsilon^-) \leq \upsilon_*$. The first step in the iterative counting process is to solve the system of equations for linear elastic constraints. The next iterations are performed only if the inequality $(u^+ - u^-) > u_*$ or $(\upsilon^+ - \upsilon^-) > \upsilon_*$ holds on a part of the pre-fracture zone. For such iterations, a system of equations is solved in each approximation for quasi-elastic bonds with a pre-fracture zone changing along the banks and depending on the magnitude of the forces in the bonds of effective compliance, which was calculated at the previous calculation step. The calculation of the effective compliance is carried out similarly to the determination of the secant modulus in the method of variable parameters of elasticity. The process of successive approximations ends when the forces along the pre-fracture zone obtained at two successive iterations practically do not differ. The nonlinear part of the bond deformation curve was approximated by a bilinear dependence [3], the ascending section of which corresponded to the deformation of the $(0 < (u^+ - u^-) \leq u_*)$ bonds with their maximum bond strength. At $(u^+ - u^-) > u_*$, the deformation law was described by a nonlinear dependence determined by points (u_*, τ_*) and (δ_c, τ_c), and at $\tau_c \geq \tau_*$ there was an increasing linear dependence (linear hardening corresponding to the elastoplastic deformation of the bonds).

To determine the ultimate equilibrium state of the medium, at which a crack appears, we use condition (5). Using the obtained solution, the conditions determining the ultimate external load were found to be as follows [8]:

$$C\left(d, q_y(d)\right) q_y(d) = \delta_{IIr}, \quad C\left(d^*, q_x(d^*)\right) q_x(d^*) = \delta_{IIr}. \qquad (36)$$

Here $x = \pm d$ and $x = \pm d^*$ are the coordinates of the points where the crack is formed, respectively.

As a result of a numerical calculation, the length of the pre-fracture zones, the forces in the bonds, and the displacement of the opposite edges of the pre-fracture zones from the loading parameter τ_{xy}^{∞} were found.

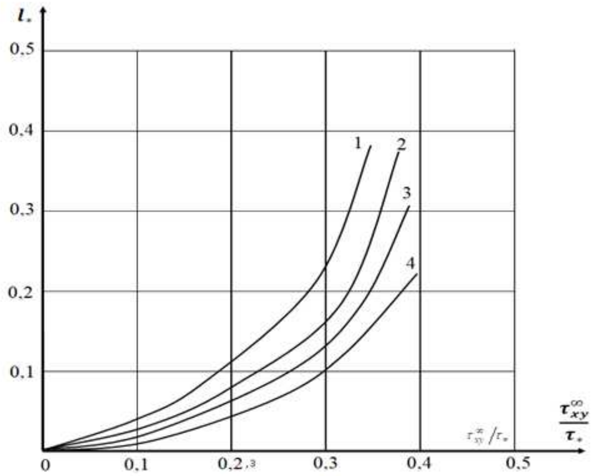


Fig. 2. Dependences of the relative length of the pre-fracture zone $\ell_* = (\ell - a)/\lambda$ on the dimensionless value of the external load $\tau_{xy}^{\infty}/\tau_*$ for some values of the radius of the holes $\lambda = 0,2 \div 0,5$ (curves 1 - 4)

Calculations were carried out to determine the ultimate loads causing crack growth. Each of the infinite systems was cut down to five equations, and with the help of one of them the unknown coefficients β_{2k} were excluded from the remaining equations. The resulting system in each approximation was solved by the Gauss method. In the calculations, h was considered constant, equal to $h = 0,90$; $r - b = 0,3$. In addition, it was adopted for the binder material $v = 0,32$; $\mu = 2,5 \cdot 10^5 MPa$, and for inclusion material $v_0 = 0,33$; $\mu_0 = 4,6 \cdot 10^5 MPa$.

In fig. 2 shows the graphs of the dependence of the relative length of the pre-fracture zone $\ell_* = (\ell - a)/\lambda$ on the dimensionless value of the external loading $\tau_{xy}^{\infty}/\tau_*$ for different values of the radius of the holes (curves 1 - 4): $1 - \lambda = 0,2$; $2 - \lambda = 0,3$; $3 - \lambda = 0,4$; $4 - \lambda = 0,5$.

In fig. 3 shows the dependence of the forces in the bonds q_x/τ_{xy}^{∞} along the pre-fracture zone on the dimensionless coordinate $x = (\ell + a)/2 + x'(\ell - a)/2$ for different values of the radius of the holes: $\lambda = 0,2 \div 0,5$ (curves $1 - 4$).

The joint solution of the resolving algebraic system and conditions (36) makes it possible (for the given characteristics of the material resistance cracks) to determine the critical value of the external load, the sizes of the pre-fracture zones for the state of limiting equilibrium at which the crack appears.

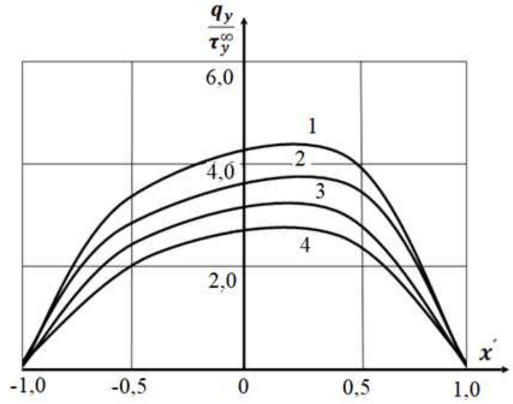


Fig. 3. Dependences of the distribution of shear stresses in the bonds q_x/τ_{xy}^{∞}along the pre-fracture zone for different values of the radius of the holes: $\lambda = 0,2 \div 0,5$ (curves 1 - 4)

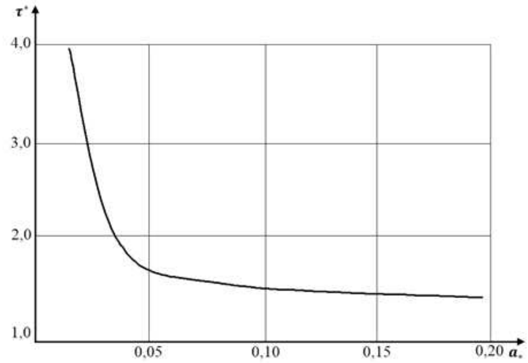


Fig. 4. Dependence of the critical load $\tau^* = \tau_{xy}^{\infty} \backslash \tau_*$ on the distance $a_* = a - \lambda$

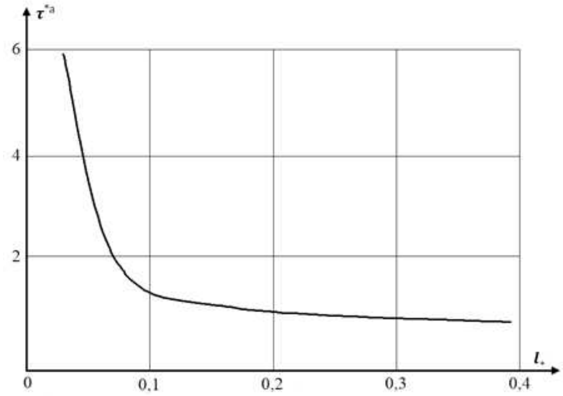


Fig. 5. Dependence of the critical load $\tau^{*a} = \tau_{xy}^{\infty}/\tau_*$ with a change in the length of the pre-fracture zone $\ell_* = \ell - a$ at $\lambda = 0,3$, $a_* = 0,05$

On the basis of the obtained numerical results in fig. 4, graphs of the dependence of the critical load $\tau^* = \tau_{xy}^{\infty} / \tau_*$ on the distance $a_* = a - \lambda$ for the pre-fracture zone collinear to the abscissa axis at $\lambda = 0,3$ are plotted.

In fig. 5 shows the dependence of the critical load τ^{*a} with a change in the length of the pre-fracture zone $\ell_* = \ell - a$ for $\lambda = 0,3$, $a_* = 0,05$.

Based on the numerical results, the graphs of the dependence of the critical (ultimate) load $\tau_*^{\ell} = \tau_{xy}^{\infty} \sqrt{\omega} / K_{IIc}$ on the crack length in the plane and in the inclusion were constructed.

In fig. 6. shows the graphs of the dependence of the critical load on the crack length in the inclusion.

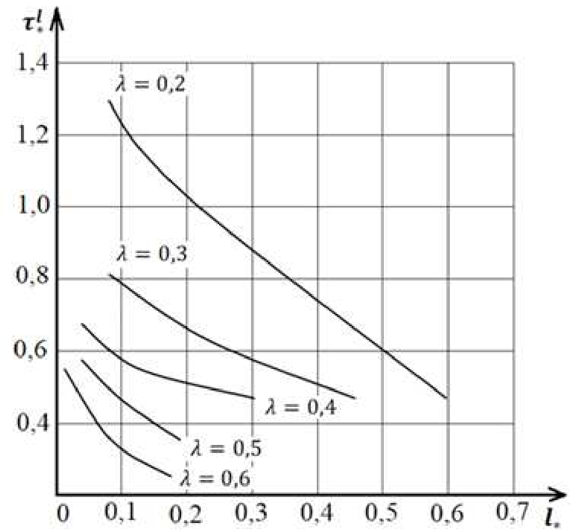


Fig. 6. Dependences of the critical load $\tau_*^{\ell} = \tau_{xy}^{\infty} \sqrt{\omega} / K_{IIc}$ on the crack length in the inclusion

6. Conclusion

The analysis of the limiting equilibrium state of a body with a doubly periodic system of rigid inclusions and the banks of the pre-fracture zones with bonds between the banks at transverse shear is reduced to a parametric study of the resolving algebraic system (24), (25), (27), (28)–(30), (31), (34) and deformation criterion of destruction (36) for different laws of deformation of interparticle bonds of the material, elastic constants and geometric characteristics of the perforated body. Directly from the solution of the obtained algebraic systems, the forces in the bonds and the displacement of the banks of the zones before destruction are determined. The model of crack initiation with bonds between the faces makes it possible to study the main regularities of the distribution of forces in the bonds at various laws of their deformation, to analyze the limiting equilibrium of the medium with the pre-fracture zone taking into account the deformation condition of crack initiation, and to estimate the critical external load and crack resistance of the material.

The relations obtained make it possible to study the limiting equilibrium state of a medium with a doubly periodic system of circular holes filled with absolutely rigid inclusions soldered along the bypass, and weakened by rectilinear pre-fracture zones with bonds between the banks of unequal length collinear to the abscissa and ordinate axes during transverse shear.

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