Baku Mathematical Journal
2022, Vol. 1, No 1, PP. 111-121
https://doi.org/10.32010/j.bmj.2022.12

ON THE ASYMPTOTICS OF THE SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR A QUASILINEAR ELLIPTIC EQUATION DEGENERATING INTO A PARABOLIC EQUATION

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Received: 27.12.2021 / Revised: 18.02.2022 / Accepted: 25.02.2022

Abstract. *In a rectangle we consider a boundary value problem for a second order quasilinear elliptic equation degenerating into a parabolic equation construct total asymptotics of the generalized solution of the problem under consideration and estimate the residual.*

Keywords: boundary value problem, elliptic equation, parabolic equation, small parameter, generalized solution, iterative process

Mathematics Subject Classification (2020): 35B25, 35C20

1. Introduction

There are a number of works devoted to the construction of the asympotics of the solution of various boundary value problems for nonlinear elliptic equations with a small parameter at higher derivatives. Note some of them [1]-[7], [9], [10]. In [1]-[3], [7] input equations are degenerate into functional ones, or into ordinary differential equations. Boundary value problems for a quasiliniear elliptic equation degenerating into a hyperbolic equation in a rectangular domain, in a curvilinear trapezoid, in a semi-infinite and finite strip were studied in [4]-[6], [9], [10].

In the present, in $D = \{(x,y)\,|0 \leq x \leq a,\ 0 \leq y \leq 1\}$ we consider the following boundary value problem

$$L_\varepsilon U \equiv -\varepsilon^p \left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^P + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)^P\right] - \varepsilon \Delta U + \frac{\partial U}{\partial x} - \frac{\partial^2 U}{\partial y^2} + cU - f(x,y) = 0, \quad (1)$$

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$$U|_{x=0} = U|_{x=a} = 0 \ \ (0 \le y \le 1)\,, \ \ U|_{y=0} = U|_{y=1} = 0 \ \ (0 \le x \le 0)\,, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $p = 2k+1$, k is an arbitrary natural number, $\Delta \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, $c > 0$ is a constant, $f(x,y)$ is a given smooth function.

In this paper our goal is to construct asymptotic expansion of the generalized solution of problem (1), (2) from the class $\overset{\circ}{W}{}^{1}_{2}(D)$. When constructing it we follow the M. I. Vishik and L. A. Lyusternik technique [11]. For constructing the asymptotics we carry out iterative processes.

2. Main Results

In the first iterative process we will look for the appropriate solution of the equation (1) in the from

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{3}$$

and the functions $W_i(x,y)$, $i = 0,1,...,n$, will be chosen in such a way that

$$L_\varepsilon W = o\left(\varepsilon^{n+1}\right). \tag{4}$$

Substituting (3) in (4), expanding nonlinear terms in powers of ε and equating the terms with the same powers of ε, for determining $W_i; i = 0,1,...,n$ we obtain the following recurrently connected equations :

$$\frac{\partial W_i}{\partial x} - \frac{\partial^2 W_i}{\partial y^2} + aW_i = f_i(x,y), i = 0,1,...,n, \tag{5}$$

where $f_0(x,y) = f(x,y), f_i(x,y)$ are the known functions dependent on $W_0, W_1, ..., W_{i-1}$, $i = 1,2,...,n$. For example, the function $f_1(x,y)$ is of the form: $f_1(x,y) = \Delta W_0$.

Equations (5) will be solved under the following boundary conditions:

$$W_i|_{x=0} = 0 \ \ (0 \le y \le 1)\,, \ W_i|_{y=0} = W_i|_{y=1} = 0 \ \ \ (0 \le x \le a)\,, i = 0,1,...,n. \tag{6}$$

The following lemma is valid.

Lemma 1. *Let $f(x,y) \in C^{n+1,2n+6}(D)$ and the condition*

$$\frac{\partial^{2k} f(x,0)}{\partial y^{2k}} = \frac{\partial^{2k} f(x,1)}{\partial y^{2k}} = 0, \ \ k = 0,1,...,n+2, \tag{7}$$

be fulfilled. Then the solution of problem (5), (6) for $i = 0$ is contained into the space $C^{n+2,2n+4}(D)$ and satisfies the relation

$$\frac{\partial^{i_1+2i_2} W_0(x,0)}{\partial x^{i_1} \partial y^{2i_2}} = \frac{\partial^{i_1+2i_2} W_0(x,1)}{\partial x^{i_1} \partial y^{2i_2}} = 0, \ i_1 + i_2 \le n+2. \tag{8}$$

Proof. Obviously, the solution of problem (5), (6) for $i = 0$ can be represented by the formula

$$W_0(x,y) = \sum_{k=1}^{\infty} \bar{W}_{0k}(x,y), \tag{9}$$

where by $\bar{W}_{0k}(x,y)$ we denoted the function

$$\bar{W}_{0k}(x,y) = \left[\int_0^x e^{-\left(c+k^2\pi^2\right)(x-\tau)} f_k(\tau)\right] \sin k\pi y, \tag{10}$$

moreover $f_k(x) = 2\int_0^1 f(x,\xi)\sin k\pi\xi d\xi$. Taking into account condition (7), we can get the estimate:

$$\left|f_k^{(i)}(x)\right| \le \frac{2M_{i,2n+4}}{k^{2n+4}\pi^{2n+4}}, \ \ i = 0,1,...,n+1, \ \ x \in [0,a], \tag{11}$$

where $M_{i,2n+4} = \max\limits_{(x,y)\in D}\left|\dfrac{\partial^{i,2n+4} f(x,y)}{\partial x^i \partial y^{2n+4}}\right|$, $i = 0,1,...,n+1$. Based on (11) from it follows from (10) that

$$\left|\frac{\partial^i \bar{\mathrm{W}}_{0\mathrm{k}}(x,y)}{\partial x^{i_1}\partial y^{i_2}}\right| \le \frac{C}{k^{2n+4-2i_1-i_2}\pi^{2n+4-2i_1-i_2}}, \ C = const, \ (x,y) \in D. \tag{12}$$

Denoting $r = 2n+4-2i_1-i_2$, from (12) we obtain that the number series $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^r}$ is majorant for the functional series, $\sum\limits_{k=1}^{\infty}\dfrac{\partial^i \bar{W}_{0k}(x,y)}{\partial x^{i_1}\partial y^{i_2}}$, obtained by term by term differentiation of (9). And this number series converges for $r \ge 2$, i.e. for $2i_1 + i_2 \le 2n+4$. This implies that W_0 belongs to the space $C^{n+2,2n+4}(D)$ and that (8) is valid. ◀

By lemma 1 the function $f_1(x,y)$ which is the right hand side of equation (5) for $i = 1$ satisfies condition (7) for $k = 0,1,...,n+1$. Then by the same lemma the function W_1 which is the solution of problem (5), (6) for $i = 1$ will satisfy condition (8) for $i_1 + i_2 \le n+1$. Continuing this process, we construct all the functions W_i, $i = 0,1,...,n$, included in the right hand side of (3).

If follows from (3) and (6) that the structured function W satisfies the following boundary conditions:

$$W|_{x=0} = 0 \ \ (0 \le y \le 1); \ \ W|_{y=0} = W|_{y=1} = 0 \ \ \ (0 \le x \le a). \tag{13}$$

This function, generally speaking, does not satisfy the boundary condition from (2) for $x = a$.

Therefore we should construct such a boundary layer type function near the boundary $x = a$ that the obtained sum $W + V$ satisfies the boundary condition

$$(W+V)|_{x=a} = 0.$$

Furthermore, when choosing V it is necessary to ensure the fulfillment of the equality

$$L_{\varepsilon,1}(W+V) - L_{\varepsilon,1}W = o\left(\varepsilon^{n+1}\right). \tag{14}$$

In (14) denotes a $L_{\varepsilon,1}$ new splitting of the operator L_ε near the boundary $x=a$. In order to write a new splitting of the operator L_ε near the boundary $x=a$, we make a change of variables: $a-x=\varepsilon\tau, y=y$. Let us consider the auxiliary function $r=\sum\limits_{j=0}^{n+1} r_j(\tau,y)$, where $r_j(\tau,y)$ are some smooth functions. The expansion of $L_\varepsilon(r)$ in powers of ε in the coordinates (τ,y) is of the form

$$L_{\varepsilon,1}r \equiv -\varepsilon^{-1}\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1}+\frac{\partial^2 r_0}{\partial\tau^2}+\frac{\partial r_0}{\partial\tau}+\right.$$

$$\left.+\sum_{j=1}^{n+1}\varepsilon^j\left[(2k+1)\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}\frac{\partial r_j}{\partial\tau}\right)+\frac{\partial^2 r_j}{\partial\tau^2}+\frac{\partial r_j}{\partial\tau}+H_j\right]+o\left(\varepsilon^{n+2}\right)\right\}, \qquad (15)$$

where $H_j(r_0,r_1,...,r_{j-1})$ are known functions dependent on $r_0,r_1,...,r_{j-1}$ and their first and second derivatives.

We look for a boundary layer type function V, near the boundary $x=a$ in the form

$$V=V_0(\tau,y)+\varepsilon V_1(\tau,y)+...+\varepsilon^{n+1}V_{n+1}(\tau,y). \qquad (16)$$

Expanding each function $W_i(a-\varepsilon\tau,y)$ at the point (a,y) by the Taylor formula, we get a new expansion of the function W in powers of ε in the coordinates (τ,y) in the form

$$W=\sum_{j=0}^{n+1}\varepsilon^j\omega_j(\tau,y)+o\left(\varepsilon^{n+2}\right). \qquad (17)$$

Here $\omega_0=\omega_0(a,y)$ is independent of τ, and the rest of the functions are determined by the formula $\omega_k=\sum\limits_{i+j=k}(-1)^i\dfrac{\partial^i W_j(a,y)}{\partial x^i}\tau^i, \quad k=1,2,...,n+1.$

Substituting the expressions (16), (17) for the functions $V,\ W$ in (14) and taking into account (15), we get the following equations for determining the functions $V_0,V_1,...,V_{n+1}$:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial V_0}{\partial\tau}\right)^{2k+1}+\frac{\partial^2 V_0}{\partial\tau^2}+\frac{\partial V_0}{\partial\tau}=0, \qquad (18)$$

$$(2k+1)\frac{\partial}{\partial\tau}\left[\left(\frac{\partial V_0}{\partial\tau}\right)^{2k}\frac{\partial V_j}{\partial\tau}\right]+\frac{\partial^2 V_j}{\partial\tau^2}+\frac{\partial V_j}{\partial\tau}=Q_j,\ j=1,2,...,n+1. \qquad (19)$$

Here Q_j are the known functions dependent on $\tau,y,V_0,V_1,...,V_{j-1},\ \omega_0,\omega_1,...,\omega_j$ and their first and second derivatives. Formulas for Q_j can be written explicitly, but they are rather bulky. We here indicate the explicit from only of the function Q_1:

$$Q_1=-(2k+1)\frac{\partial}{\partial\tau}\left(\left(\frac{\partial V_0}{\partial\tau}\right)^{2k}\frac{\partial\omega_1}{\partial\tau}\right)-\frac{\partial^2 V_0}{\partial y^2}+CV_0.$$

The boundary conditions for the equations (18), (19) are obtained from the requirement that the sum $W+V$ satisfies the boundary condition

$$(W+V)|_{x=a}=0. \tag{20}$$

Substituting the expressions for W in from (3) and for V from (16) in (20) and taking into account that we are looking for $V_j,\ j=0,1,...,n+1$ as a boundary layer type function, we have

$$V_0|_{\tau=0}=\varphi_0\left(y\right), \quad \lim_{\tau\to+\infty}V_0=0, \tag{21}$$

$$V_j|_{\tau=0}=\varphi_j\left(y\right), \quad \lim_{\tau\to+\infty}V_j=0, \qquad j=1,2,...,n+1, \tag{22}$$

where $\varphi_i\left(y\right)=-W_i\left(1,y\right)$ for $i=0,1,..,n;\ \varphi_{n+1}\equiv 0.$

The following lemma is valid.

Lemma 2. *For every $y\in[0,1]$ the problem (18), (21) has a unique solution that is infinitely differentiable with respect to τ, and with respect to y has continuous derivatives to the $(2n+4)$-th order inclusively. Therewith the estimation*

$$\left|\frac{\partial^i V_0(\tau,y)}{\partial\tau^{i_1}\partial y^{i_2}}\right|\le g_i\left(\left|\varphi_0(y)\right|,\left|\varphi_0'(y)\right|,....,\left|\varphi_0^{(i_2)}(y)\right|\right)e^{-\tau}, \tag{23}$$

is valid, where $i=i_1+i_2;\quad i_2=0,1,...,2n+4; g_i\left(t_1,t_2,...,t_{i_2+1}\right)$ are some known polynomials of their own arguments with non-negative coefficients, and free terms of these polynomials equal zero, and even at least one of other coefficients is non-zero.

Proof. The existence and uniqueness of the solution of problem (18), (21) was proved in [8]. The solution of the problem (18), (21) for $y=0$ and $y=1$ is redefined by an identity zero, and for $y\in(0,1)$ the solution in the parametric form is represented by the following formulas:

$$\tau=\frac{2k+1}{2k}\left(q_0^{2k}-q^{2k}\right)+\ln\left|\frac{q_0}{q}\right|, V_0=-q^{2k+1}-q. \tag{24}$$

Here q is a parameter, $q_0\left(y\right)$ is a real root of the algebraic equation

$$q_0^{2k+1}+q_0+\varphi_0(y)=0. \tag{25}$$

The smoothness of the solution of the problem (18), (21) was also proved in [8]. Therefore, here we derive only the estimation (23). From the first equality of (24) one can obtain the estimation

$$|q|\le|q_0(y)|\exp\left[\frac{2k+1}{2k}q_0^{2k}\left(y\right)\right]\exp\left(-\tau\right). \tag{26}$$

Transforming the equation (25) we have: $q_0(y)=\left[q_0^{2k}(y)+1\right]^{-1}\varphi_0(y)$, hence if follows $|q_0(y)|\le|\varphi_0(y)|$. Hence we have that $\exp\left[\frac{2k+1}{2k}q_0^{2k}\left(y\right)\right]$ is bounded, i.e. $\exp\left[\frac{2k+1}{2k}q_0^{2k}\left(y\right)\right]\le C_0$. Consequently, from (26) we get the estimation

$$|q|\le C_0\left|\varphi_0(y)\right|\exp\left(-\tau\right). \tag{27}$$

Taking into account (27) in the second equality of (24) we have

$$|V_0| \le C\,|\varphi_0(y)|\exp(-\tau)\,,\ C>0. \tag{28}$$

Recalling that the parametric from (24) of the solution of the problem (18), (21) was obtained by means of the substitution $\dfrac{\partial V_0}{\partial\tau}=q$, from (27) we obtain an estimation for $\dfrac{\partial V_0}{\partial\tau}$

$$\left|\frac{\partial V_0}{\partial\tau}\right| \le C_0\,|\varphi_0(y)|\exp(-\tau)\,. \tag{29}$$

The function $\dfrac{\partial^2 V_0}{\partial\tau^2}$ can be represented in the form

$$\frac{\partial^2 V_0}{\partial\tau^2} = -B^{-1}(\tau,y)\,\frac{\partial V_0}{\partial\tau}, \tag{30}$$

where $B(\tau,y)$ denotes the function

$$B(\tau,y) = (2k+1)\left(\frac{\partial V_0}{\partial\tau}\right)^{2k}+1. \tag{31}$$

Considering that $0<B^{-1}(\tau,y)\le 1$ from (29) and (30) we get an estimation for $\dfrac{\partial^2 V_0}{\partial\tau^2}$. The estimations for the derivatives $V_0(\tau,y)$ with respect to τ of higher orders can be obtained by differentiating the both hand sides of (30) with respect to τ and each time considering the estimations for previous derivatives. These estimations will be of the form (29), i.e.

$$\left|\frac{\partial^i V_0}{\partial\tau^i}\right| \le C_0\,|\varphi_0(y)|\exp(-\tau)\,, i=2,3,\dots\,. \tag{32}$$

We pass to the proof of the estimations for the derivatives $V_0(\tau,y)$ with respect to y and for mixed derivatives. The function $\dfrac{\partial V_0}{\partial y}=z$ satisfies the equation in variations obtained from the equation (18) by differentiation with respect to y:

$$\frac{\partial}{\partial\tau}\left[B(\tau,y)\,\frac{\partial z}{\partial\tau}\right]+\frac{\partial z}{\partial\tau}=0. \tag{33}$$

From (21) we obtain that the function z should satisfy the boundary conditions

$$z|_{\tau=0}=\varphi_0'(y),\qquad \lim_{\tau\to+\infty} z=0. \tag{34}$$

The solution of the problem (33), (34) is of the form

$$z=\varphi_0'(y)\exp\left[-\int_0^\tau B^{-1}(\xi,y)\,d\xi\right]. \tag{35}$$

Using (31) and estimation (29) in (35) we get the estimation

$$|z| = \left| \frac{\partial V_0}{\partial y} \right| \leq C \left| \varphi_0' (y) \right| \exp(-\tau). \tag{36}$$

It follows from (35) that $\dfrac{\partial z}{\partial \tau} = -B^{-1}(\tau, y)\, z$. Considering (36), hence we obtain an estimation for the mixed derivative

$$\left| \frac{\partial z}{\partial \tau} \right| = \left| \frac{\partial^2 V_0}{\partial y \partial \tau} \right| \leq C \left| \varphi_0'(y) \right| \exp(-\tau). \tag{37}$$

We now can obtain an estimation for $\dfrac{\partial^2 V_0}{\partial y^2}$. Differentiating the both hand sides of (35) with respect to , we have

$$\frac{\partial z}{\partial y} = -\left\{ -\int_0^{\tau} \left[B^{-1}(\xi, y) \right]'_{y} d\xi \right\} z + \varphi_0''(y) \exp\left[-\int_0^{\tau} B^{-1}(\xi, y)\, d\xi \right]. \tag{38}$$

It follows from (31)

$$\left[B^{-1}(\tau, y) \right]'_{y} = -(2k+1)(2k)\, B^{-2}(\tau, y) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k-1} \frac{\partial^2 V_0}{\partial y \partial \tau}.$$

Obviously $0 < B^{-i} \leq 1$ for any natural number i. Knowing the estimation for (29) $\dfrac{\partial V_0}{\partial \tau}$ and estimation (37) for $\dfrac{\partial^2 V_0}{\partial y \partial \tau}$, we estimate $\left[B^{-1}(\tau, y) \right]'_{y}$:

$$\left| \left[B^{-1}(\tau, y) \right]'_{y} \right| \leq C \left| \varphi_0(y) \right| \left| \varphi_0'(y) \right| \exp(-\tau). \tag{39}$$

Taking into account (36) and (39) in (38), we have

$$\left| \frac{\partial z}{\partial y} \right| = \left| \frac{\partial^2 V_0}{\partial y^2} \right| \leq \left[C_1 \left| \varphi_0(y) \right| \left| \varphi_0'(y) \right|^2 + C_2 \left| \varphi_0''(y) \right| \right] \exp(-\tau). \tag{40}$$

In the same way we prove the validity of the estimate (23) for the subsequent derivatives $V_0(\tau, y)$.

Lemma 2 is proved. ◀

It follows from (8) that the function $\varphi_0(y)$ and all its even derivatives vanish for $y = 0$. Hence and from the estimations that were obtained in the proof of lemma 2, it follows that the function $V_0(\tau, y)$, all its derivatives with respect to τ and all even derivatives with respect to y vanish for $y = 0$ (see.(29), (32), (40)).

Lemma 3. *The problems (19), (22) have unique solutions and the functions $V_j(\tau, y)$, $j = 1, 2, ..., n+1$, with respect to τ are infinitely differentiable, and with respect to y*

have continuous derivatives to the $(2n+2-2j)$*-th order inclusively. And the following estimations of the from*

$$\left|\frac{\partial^i V_j(\tau,y)}{\partial\tau^{i_1}\partial y^{i_2}}\right| \le \left[\sum_{S=0}^{i_2+j+1} |q_{js}(y)|\,\tau^s\right]\exp(-\tau)\,, \tag{41}$$

are valid, where $i_2 = 0,1,...,2n+2-j$, $j = 1,2,...,n+1$, $q_{js}(y)$ *are the known functions.*

Proof. In [8] the existence, uniqueness and smoothness of the solution of problems (19), (22) is proved, and the representation of these solutions is obtained in the following way:

$$V_j(\tau,y) = \left\{\varphi_j(y) - \int_0^\tau \left[B^{-1}(z,y)\,e^{\nu(z,y)}\int_z^{+\infty} Q_j(\xi,y)\,d\xi\right]dz\right\}\exp\left[-\nu(\tau,y)\right]. \tag{42}$$

Here, $\nu(\tau,y)$ denotes the function

$$\nu(\tau,y) = \int_0^\tau B^{-1}(\xi,y)\,d\xi.$$

Substituting $j=1$ in (42), we obtain a formula for $V_1(\tau,y)$. Using explicit expressions $Q_1(\tau,y)\,,\omega_1(\tau,y)$, and considering known estimations for V_0, $\frac{\partial V_0}{\partial\tau}$, $\frac{\partial^2 V_0}{\partial y^2}$, we obtain

$$|Q_1(\tau,y)| \le |q_1(y)|\exp(-\tau), \tag{43}$$

where $q_1(y)$ is a known function and moreover $q_1^{(2k)}(0) = q_1^{(2k)}(1) = 0$, $k = 0,1,...,n+1$. Following (43), from (42) (for $j=1$) we can obtain the following estimation:

$$|V_1(\tau,y)| \le C\left(|\varphi_1(y)| + \tau\,|q_1(y)|\right)\exp(-\tau). \tag{44}$$

Differentiating the both hand sides of (42) (for $j=1$) with respect to τ, we have

$$\frac{\partial V_1}{\partial\tau} = -B^{-1}(\tau,y)\left[V_1 + \int_\tau^{+\infty} Q_1(\xi,y)\,d\xi\right]. \tag{45}$$

Using the estimations (43) and (44) in (45), we obtain an estimation for $\frac{\partial V_1}{\partial\tau}$. The estimation for higher derivatives with respect to τ are obtained from the formulas obtained by successive differentiation of both hand sides of (45) and from the estimations for pervious derivatives $V_1(\tau,y)$. Note that threes estimations are of the form

$$\left|\frac{\partial^i V_1(\tau,y)}{\partial\tau^i}\right| \le \left(|q_1(y)| + |q_2(y)|\,\tau\right)\exp(-\tau),\;\; i = 1,2,...\,,$$

where $q_2(y)$ is a known function, and $q_2^{(2k)}(0) = q_2^{(2k)}(1) = 0$, $\;k = 0,1,...,n+1$.

We now derive estimations for the derivatives $V_1(\tau,y)$ with respect to y and for mixed derivatives. The function $\frac{\partial V_1}{\partial y}$ can be defined as the solution of a boundary value problem for an equation in variations that is obtained from (19) (for $j=1$) by differentiation with respect to y. One can notice that the function $\dfrac{\partial V_1}{\partial y}$ also is defined by formula (42), but in it the function $\varphi_j(y)$ should be replaced by $\varphi_1'(y)$, the function $\int_z^{+\infty} Q_j(\xi, y)\, d\xi$ by the following function:

$$\int_z^{+\infty} Q_{1y}'(\xi, y)\, d\xi + B_y'(z, y)\, \frac{\partial V_1(z, y)}{\partial z}.$$

Consequently this time when obtaining estimations, instead of (43), we have to use the estimation

$$\left| \int_z^{+\infty} Q_{1y}(\xi, y)\, d\xi + B_y'(z, y)\, \frac{\partial V_1(z, y)}{\partial z} \right| \le (|q_1(y)| + |q_2(y)|\, z) \exp(-z).$$

As a result, we obtain an estimation for $\dfrac{\partial V_1}{\partial y}$ in the form

$$\left| \frac{\partial V_1}{\partial y} \right| \le \left(|q_1(y)| + |q_2(y)|\, \tau + |q_3(y)|\, \tau^2 \right) \exp(-\tau). \tag{46}$$

If to differentiate the both hand sides of the formula for $\dfrac{\partial V_1}{\partial y}$ with respect to τ, we again obtain an estimation of the form (46). It should be noted that for each differentiation of $V_1(\tau, y)$ with respect to τ, the degree of the polynomial with respect to $V_1(\tau, y)$ standing in the right hand side of the estimation increases by one unit. The estimation for $V_1(\tau, y)$ in the general case is of the form

$$\left| \frac{\partial^i V_1(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \le \left(|q_{10}(y)| + |q_{11}(y)|\, \tau + ... + |q_{1 i_2+1}(y)|\, \tau^{i_2+1} \right) \exp(-\tau).$$

Continuing this process and considering each time the exploit form of the right hand side of the equation for V_j, we obtain the estimation (41).

Lemma 3 is proved. ◄

We multiply all the functions $V_j,\ j = 0, 1, ..., n+1$, by a smoothing factor and for the obtained new functions we leave previous denotation. An the expense of smoothing factors all the functions $V_j,\ j = 0, 1, ..., n+1$, vanish for $x = 0$. Therefore, hence and from (13) it follows that the constructed sum $W + V$, in addition to the boundary condition (20), satisfies also the condition

$$(W + V)|_{x=0} = 0. \tag{47}$$

It is known from the construction process that all the functions $V_j(\tau, y)\, , j = 0, 1, ..., n+1$, vanish for $y = 0$ and $y = 1$. Hence and from (13) if follows that the sum $W + V$ alongside with the conditions (20), (47) satisfies the following boundary conditions as well:

$$(W + V)|_{y=0} = 0, \ \ (W + V)|_{y=1} = 0. \tag{48}$$

Thus, the constructed sum $\tilde{U} = W + V$ satisfies the boundary conditions (20), (47) and (48). Having denoted $U - \tilde{U} = z$, we have the following asymptotic expansion in a small parameter of the solution of the problem (1), (2):

$$U = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + z, \tag{49}$$

where z is a residual.

We have the following lemma.

Lemma 4. *For the residual z the following estimation is valid*

$$\varepsilon^p \iint\limits_D \left[\left(\frac{\partial z}{\partial x} \right)^p + \left(\frac{\partial z}{\partial y} \right)^p \right] dxdy + \varepsilon \iint\limits_D \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dxdy +$$

$$+ \iint\limits_D \left(\frac{\partial z}{\partial y} \right)^2 dxdy + C_1 \iint\limits_D z^2 dxdy \le C_2 \varepsilon^{2(n+1)}, \tag{50}$$

where $C_1 > 0, C_2 > 0$ are constants independent of ε.

Proof. Adding (4) and (14), we have::

$$L_\varepsilon \left(\tilde{U} \right) = o \left(\varepsilon^{n+1} \right). \tag{51}$$

Subtracting the equation (51) from (1), we obtain

$$-\varepsilon^p \left\{ \frac{\partial}{\partial x} \left[\left(\frac{\partial U}{\partial x} \right)^p - \left(\frac{\partial \tilde{U}}{\partial x} \right)^p \right] + \frac{\partial}{\partial y} \left[\left(\frac{\partial U}{\partial y} \right)^p - \left(\frac{\partial \tilde{U}}{\partial y} \right)^p \right] \right\} -$$

$$-\varepsilon \Delta z + \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} + cz = \varepsilon^{n+1} F(\varepsilon, x, y), \tag{52}$$

where $\|F(\varepsilon, x, y)\|_{L_2(D)} \le C$ for any $\varepsilon \in [0, \varepsilon_0)$, and $C > 0$ is independent of ε.

If follows from (2), (20), (47), (48) and (49) that z satisfies the boundary conditions

$$z|_{x=0} = z|_{x=a} = 0, \quad z|_{y=0} = z|_{y=1} = 0. \tag{53}$$

Multiplying the both hand sides of (52) by $z = U - \tilde{U}$, and integrating by parts allowing for boundary conditions (53), after certain transformations we get the estimation (50).

Lemma 4 is proved. ◄

Combining the obtained results, we arrive at the following statement.

Theorem. *Let $f(x, y) \in C^{n+1,2n+6}(D)$ and condition (7) be fulfilled. Then for the generalized solution of the problem (1), (2) we have the asymptotic expansion, (50) where the functions W_i are defined by the first iterative process, V_j are boundary layer type functions near the boundary $x = 1$, and z is a residual, moreover the estimation (50) is valid for it (49).*

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