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APPROXIMATE SOLUTION OF OPTIMAL CONTROL PROBLEM FOR COOLING PROCESS WITH MINIMUM ENERGY

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Abstract. *In the paper, optimal control problem for cooling process with minimum energy in materials with heat conducting viscosity is considered. To solve approximately the considered problem, a finite-dimensional approximation for the solution of the corresponding boundary value problem in the form of truncated Fourier series is constructed and an integral representation for the coefficients of this series is obtained. This yields a system of integral equations with respect to control parameters. So, the problem is reduced to finding a minimum norm function from these moment relations. Applying the theorem on orthogonal decomposition of a normalized space, every approximation of control parameter and the corresponding value of a functional in analytic form is finded.*

Keywords: optimal control, cooling process, approximate solution, Fourier method, the problem of moments

Mathematics Subject Classification (2020): 49K20, 49J20, 93D15

1. Problem Statement

The temperature gradient in the materials with heat conducting viscosity is varying in the direction of heat transfer. Therefore, the cooling process in such materials is described by the highest order equation of parabolic type. In other words, this process is described by the function $u(x,t)$ which satisfies the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + \xi \frac{\partial^3 u}{\partial t \partial x^2} + p(x,t), \tag{1}$$

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inside the domain $\overline{Q} = \{0 \le x \le 1,\ 0 \le t \le T\}$, and the initial and boundary conditions

$$u(x,0) = u^0(x), \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(1,t)}{\partial x} + \alpha u(1,t) = 0, \tag{3}$$

where a, $\alpha > 0$ are real constants, $\xi = const > 0$ is a coefficient of heat conducting viscosity, $u^0(x) \in L_2(0,1)$ is an initial temperature distribution, and $p(x,t) \in L_2(Q)$ is an internal heat source, which plays the role of a control parameter.

Let $\varphi(x) \in L_2(0,1)$ be a given function. Our optimal control problem is to find a control function such that the corresponding solution of the problem (1)–(3) satisfies the condition

$$u(x,T) = \varphi(x),$$

and the functional

$$I[p] = \iint\limits_{Q} p^2(x,t)dxdt, \tag{4}$$

takes the least possible value.

In [1], [3], the optimal synthesis control problem for this process has been solved in cases where the control parameter is distributed and concentrated. In [4], the finite-dimensional approximation of the solution of this boundary value problem corresponding to the admissible control has been constructed.

2. Applying l-Problem of Moments

Consider in $L_2(0,1)$ the orthonormal system of functions $X_n(x) = \frac{\cos \lambda_n x}{\sqrt{\omega_n}}$, $n = 1,2,\ldots$, where λ_n's are the eigenvalues of the boundary value problem

$$X^{"}(x) + \lambda^2 X(x) = 0, \quad 0 < x < 1; \quad X^{'} = 0, \quad X^{'}(1) + \alpha X(1) = 0; \tag{5}$$

which are the positive roots of the equation $\lambda \tan \lambda = \alpha$, and $\omega_n = \frac{\alpha + \alpha^2 + \lambda_n^2}{2(\alpha^2 + \lambda_n^2)}$ is a normalizing factor. With this in mind, we will seek for the approximate solution of the problem (1)–(3) in the form of truncated Fourier series

$$u^N(x,t) = \sum_{n=1}^{N} u_n^N(t) X_n(x), \quad u_n^N(t) = \int\limits_0^1 u_n^N(x,t) X_n(x) dx. \tag{6}$$

Lets multiply both sides of the equation (1) by $X_n(x)$ and integrate with respect to x from 0 to 1. Taking into account the boundary conditions (3) and (5), we see that the coefficients $u_n^N(t)$ satisfy the following system of ordinary differential equations [3]:

$$\frac{du_n^N}{dt} = -\frac{a^2 \lambda_k^2}{1 + \xi \lambda_k^2} u_n^N(t) + \frac{1}{1 + \xi \lambda_k^2} p_n(t), \quad n = 1,2,\ldots,N. \tag{7}$$

As the function (6) must satisfy the condition (2), we have

$$u_n^N(0) = u_n^0, \quad n = 1, 2, \ldots, N, \tag{8}$$

where $p_n(t)$ and u_n^0 are the Fourier coefficients of the functions $p(t,x)$ and $u_n^0(x)$, respectively.

Then the functional (4) becomes

$$I[\overline{p}] = \sum_{n=1}^{N} I_n[p], \quad I_n[p] = \int_0^T p_n^2(t)dt. \tag{9}$$

As the Fourier coefficients $u_n^N(t)$ are defined by the system (7) independently of each other, it suffices to consider the minimization problem for the functional $I_n[\overline{p}]$.

Thus, our problem is reduced to the determination of vector functions $\overline{p}_N(t) = (p_1(t), p_2(t), \ldots, p_N(t)) \in L_2^N(0,T)$ such that the corresponding solutions of the problem (7), (8) satisfy the conditions

$$u_n^N(T) = \varphi_n, \quad n = 1, 2, \ldots, N, \tag{10}$$

and for every fixed n the functional $I_n[\overline{p}]$ takes the least possible value, where φ_n's are the Fourier coefficients of the function $\varphi(x)$.

The formulation of the considered problem in terms of the moment's problem allows one to obtain its full solution. At that, the following theorem is valid.

Theorem. *Let the controllable process be described by the boundary-value problem (1)-(3), where the admissible controls are arbitrary functions $p = p(x,t) \in L_2(Q)$ and $\varphi(x) \in L_2(0,1)$. Then the controllable energy problem (7)-(10) has a unique solution and this solution is represented in the form:*

$$p_n(t) = \frac{2a^2\nu_n^2\psi_n}{1 - e^{-a^2\nu_n^2 T}} e^{-a^2\nu_n^2(T-t)}, \ n = 1, 2, ..., N,$$

where $\psi_n = (\varphi_n - u_n^0 e^{-a^2\nu_n^2 T})(1 + \xi\lambda_n^2)$, $\nu_n^2 = \frac{a^2\lambda_n^2}{1+\xi\lambda_n^2}$, but $p_n(t)$ and φ_n are the Fourier coefficients of the functions $p(x,t)$ and $\varphi(x)$.

Proof. By Cauchy formula, the solution of the equation (7) with the initial condition (8) can be represented as follows:

$$u_n^N(T) = u_n^0 e^{-a^2\nu_n^2 t} + \frac{1}{\nu_n} \int_0^t p_n(\tau) e^{-a^2\nu_n^2(t-\tau)} d\tau, \quad n = 1, 2, \ldots, N,$$

where $\nu_n^2 = \frac{a^2\lambda_n^2}{1+\xi\lambda_n^2}$.

Then the condition (10) can be rewritten in the following form:

$$u_n^0 e^{-a^2\nu_n^2 T} + \frac{1}{1+\xi\lambda_n^2} \int_0^T p_n(t) e^{-a^2\nu_n^2(T-t)} dt = \varphi_n, \quad n = 1, 2, \ldots, N. \tag{11}$$

So we obtain the following system of equations:

$$\int_0^T p_n(t)e^{-a^2\nu_n^2(T-t)}dt = \psi_n, \quad n = 1, 2, \ldots, N, \tag{12}$$

where

$$\psi_n = \left(\varphi_n - u_n^0 e^{-a^2\nu_n^2 T}\right)\left(1 + \xi\lambda_n^2\right).$$

Thus, every admissible control which provides the satisfaction of the condition (11), definitely satisfies the moment relations (12).

Consequently, we have to find a function $p_n(t)$ with minimum norm satisfying the condition (12). Denote by H a subspace of the space $L_2(0, T)$, consisting of the elements of the form

$$q_n(t) = \alpha_n e_n(t),$$

where α_n is an arbitrary real coefficient and $e_n(t) = e^{-a^2\nu_n^2(T-t)}$. Then, by Levis theorem, every element can be uniquely represented in the form [2]

$$p_n(t) = q_n(t) + g_n(t), \quad q_n(t) \in H, \quad g_n(t) \perp H,$$

and

$$\|p_n(t)\| = \|q_n(t)\| + \|g_n(t)\|.$$

Hence it follows that

$$\int_0^T p_n^2(t)dt = \int_0^T q_n^2(t)dt, \tag{13}$$

where the component $q_n(t)$ of the element $p_n(t)$ does not affect whether this element satisfies the condition (12). Nevertheless, from (13) it follows that the component $g_n(t)$ does not affect the norm of the element $p_n(t)$. So, if the considered optimal control problem has a solution, then it belongs to H, i.e. it can be represented in the form

$$p_n(t) = \alpha_n e_n(t) \text{ or } p_n(t) = \alpha_n e^{-a^2\nu_n^2(T-t)}.$$

To determine the unknown coefficient, we substitute the obtained value into (12). Then we obtain

$$\alpha_n = \frac{2a^2\nu_n^2\psi_n}{1 - e^{-a^2\nu_n^2 T}},$$

and

$$p_n(t) = \frac{2a^2\nu_n^2\psi_n}{1 - e^{-a^2\nu_n^2 T}} e^{-a^2\nu_n^2(T-t)}.$$

The convergence of the chosen approximation is proved as in [5]. ◄

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