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BASICITY OF A PERTURBED SYSTEM OF EXPONENTS IN LEBESGUE SPACES WITH A VARIABLE SUMMABILITY EXPONENT

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Abstract. *In this paper a system of exponents* $1\bigcup\left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$ *is considered, where* $\lambda_n=\sqrt[m]{|P_m(n)|}$, $P_m(n)=n^m+\alpha_{m-1}n^{m-1}+...+\alpha_0$ *is some polynomial of degree* m, $m\in N$. *It is proved that under certain conditions on the exponent* $p(\cdot)$ *the basicity of this system in a Lebesgue space with a variable summability exponent* $L_{p(\cdot)}(-\pi,\pi)$ *depends on the coefficient* α_{m-1} *and* m. *Moreover, in the case of basicity, it is isomorphic to the classical system of exponents* $\left\{e^{int}\right\}_{n\in Z}$ *in* $L_{p(\cdot)}(-\pi,\pi)$. *Earlier in the case* $p(\cdot)\equiv m=2$, $\alpha_1=0$, *the Riesz basicity of this system in* $L_2(-\pi,\pi)$ *was established by Yu.A. Kazmin.*

Keywords: system of exponents, basicity, variable summability exponent

Mathematics Subject Classification (2020): 30B60, 42C15, 46B15

1. Introduction

The study of the basis properties (completeness, minimality, basicity) of a system of exponents of the form $E_\lambda\equiv\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ in Lebesgue spaces $L_p(a,b)$, $1\le p\le\infty$ ($L_\infty(a,b)\equiv C[a,b]$) has a very rich and long history starting with the well-known results of Paley-Wiener [21] and N. Levinson [16]. In [21], it was proved that for $\sup\limits_n|\lambda_n-n|<\frac{\ln 2}{\pi^2}$, the system E_λ forms a Riesz basis for $L_2(-\pi,\pi)$ and the question of refining the constant $\frac{\ln 2}{\pi^2}$ in this inequality was also raised there. The best constant was found by M.I. Kadets [15] and the corresponding result is known as the "$\frac{1}{4}$-Kadets" theorem. When $\{\lambda_n\}$ has the form $\lambda_n=n+\alpha\, sign\, n$, $n\in Z$, the criterion for the basicity of the system E_λ in $L_p(-\pi,\pi)$, $1<p<\infty$, was found in the work of A.M. Sedletskii [23]. The same result, including for systems of sines and cosines, was obtained in the works of E.I. Moiseev [17], [18]. These results were carried over to the complex case of a parameter α in the works of G.G. Devdariani [13], [14]. Subsequently, these results were generalized in the works

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of B.T. Bilalov [2]-[5], [10], [11]. In the work of S.R. Sadigova & A.E. Guliyeva [22] the basicity of the system E_λ (case $\lambda_n = n + \alpha\, sign\, n$) is established in a weighted space $L_{p,w}(-\pi,\pi)$, $1<p<\infty$ with a weight $w(\cdot)$ from the Muckenhoupt class $A_p(-\pi,\pi)$. A criterion for the basicity of the same system in Morrey-type spaces was found in the work of B.T. Bilalov [7] (see also [9]).

For a Lebesgue space with a variable summability exponent, a similar result was obtained in [8], the weighted case of the space was considered in [19], [20].

In this paper a system of exponents $1\bigcup\left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$ is considered, where $\lambda_n = \sqrt[m]{|P_m(n)|}$, $P_m(n) = n^m + \alpha_{m-1}n^{m-1} + ... + \alpha_0$ is some polynomial of degree m, $m\in N$. It is proved that under certain conditions on the exponent $p(\cdot)$ the basicity of this system in a Lebesgue space with a variable summability exponent $L_{p(\cdot)}(-\pi,\pi)$ depends on the coefficient α_{m-1} and m. Moreover, in the case of basicity, it is isomorphic to the classical system of exponents $\left\{e^{int}\right\}_{n\in Z}$ in $L_{p(\cdot)}(-\pi,\pi)$. Earlier in the case $p(\cdot)\equiv m=2$, $\alpha_1=0$, the Riesz basicity of this system in $L_2(-\pi,\pi)$ was established by Yu.A. Kazmin.

2. Needful Information

We will use the usual notations: N will be a set of all positive integers; $Z_+ = \{0\}\bigcup N$; Z will be a set of all integers; C will stand for the field of complex numbers; $L[\cdot]$ will be a linear span; $\bar{M}$ will be a closure of the set M; $KerT$ will be a kernel of the operator T; R_T will be a range of the operator T; $[X]$ is an algebra of bounded operators in X; $\dim M$ dimension of M; $\dot{+}$ is a direct sum; X^* is a dual space to X; T^* is conjugate to T operator; $X/_M$ is a quotient space of a space X in the subspace M; B-space is a Banach space ; $\exists!$ there exists a unique; $p' : \frac{1}{p}+\frac{1}{p'}=1$ is the conjugate number to p.

We will use the concept of a "double" basis in a Banach space X.

Definition 1. *The system $\{x_n^+; x_n^-\}_{n\in N}\subset X$ is called a double basis (or simply a basis) in the B-space X, if $\forall x\in X$; $\exists!\,\{\lambda_n^+;\lambda_n^-\}_{n\in N}\subset C$:*

$$\left\|\sum_{k=1}^{n_1}\lambda_k^+x_k^+ + \sum_{k=1}^{n_2}\lambda_k^-x_k^- - x\right\|_X \to 0\,,\ n_1,n_2\to\infty.$$

We also need some concepts and facts from the theory of close bases.

Definition 2. *The systems $\{\varphi_n\}_{n\in N}$ and $\{\psi_n\}_{n\in N}\subset X$ in B-space X are said to be p-close if*

$$\sum_n \|\varphi_n-\psi_n\|_X^p < +\infty.$$

Let us define the concept of a p-Bessel system.

Definition 3. *A minimal system* $\{x_n\}_{n\in N}\subset X$ *in a B-space* X *with conjugate system* $\{x_n^*\}_{n\in N}\subset X^*$ *is called p-Besselian if*

$$\left(\sum_n |x_n^*(f)|^p\right)^{\frac{1}{p}} \leq M\,\|f\|_X\ ,\ \forall f\in X.$$

The following theorem is true.

Theorem 1. *[6] Let p-Besselian system* $\{x_n\}_{n\in N}\subset X$ *form a basis for B-space* X *and the system* $\{y_n\}_{n\in N}\subset X$ *be a* p'*-close to* $\{x_n\}_{n\in N}$*. Then the following properties of the system* $\{y_n\}_{n\in N}\subset X$ *in* X *are equivalent: i)* $\{y_n\}_{n\in N}$ *is complete; ii)* $\{y_n\}_{n\in N}$ *is minimal; iii)* $\{y_n\}_{n\in N}$ ω*-linearly independent; iv)* $\{y_n\}_{n\in N}$ *forms a basis isomorphic to* $\{x_n\}_{n\in N}$*.*

Let us recall the definition of ω-linear independence.

Definition 4. *The system* $\{x_n\}_{n\in N}\subset X$ *is called* ω*-linearly independent in B-space* X *if it follows from* $\sum_{n=1}^{\infty}\lambda_n x_n=0$ *that* $\lambda_n=0,\ \forall n\in N$*.*

More details of these and other facts can be found, for example, from the monograph [6].

We also accept the following

Definition 5. *A system* $\{x_n\}_{n\in N}\subset X$ *in B-spaces* X *is called defective if, after adding to it and eliminating a finite number of elements from it, it becomes complete and minimal in* X*.*

We will need the following theorem from the monograph [25, p. 129].

Theorem 2. *[25] The system of exponents* $\left\{e^{i\lambda_n t}\right\}$ *is complete in* $C[a,b]$ *if and only if its closed linear span contains on other exponential function* $e^{i\lambda t}$*.*

Now we give the definition of a Lebesgue space $L_{p(\cdot)}(-\pi,\pi)$ with a variable summability exponent $p(\cdot)$. Let $p:[-\pi,\pi]\to[1,+\infty)$ be some Lebesgue measurable function. By $p:[-\pi,\pi]\to[1,+\infty)$ denote the class of all functions measurable on $[-\pi,\pi]$ (with respect to Lebesgue measure). Denote

$$I_{p(\cdot)}(f)=\int_{-\pi}^{\pi}|f(t)|^{p(t)}\,dt.$$

Let

$$L_{p(\cdot)}(-\pi,\pi)=\left\{f\in \mathrm{L}_0:\ I_{p(\cdot)}(f)<+\infty\right\}.$$

Assume

$$p^-=\inf_{(-\pi,\pi)} vrai\ p(t)\,;\quad p^+=\sup_{(-\pi,\pi)} vrai\ p(t)\,.$$

For $p^+ < +\infty$, $L_{p(\cdot)}(-\pi,\pi)$ is a linear space and moreover with respect to the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$L_{p(\cdot)}(-\pi,\pi)$ is a Banach space.

Let us introduce the following class of functions $p(\cdot)$:

$$WL(-\pi,\pi) = \{p(\cdot) : p(-\pi) = p(\pi) \quad \& \quad \exists c > 0 :$$

$$\forall t_1, t_2 \in [-\pi,\pi] : |t_1 - t_2| \le \frac{1}{2} \Rightarrow |p(t_1) - p(t_2)| \le \frac{c}{-\ln|t_1 - t_2|}\right\}.$$

The following property is known.

Property. *[12] If $p(\cdot) : 1 < p^- \le p^+ < +\infty$, then the class of functions $C_0^\infty(-\pi,\pi)$ (compactly supported and infinitely differentiable) is everywhere dense in $L_{p(\cdot)}(-\pi,\pi)$.*

By $p'(\cdot) : \frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ we will denote the conjugate of a function $p(\cdot)$. The following generalized Hölder inequality is true.

Statement 1. *Let $1 < p^- \le p^+ < +\infty$. Then $\exists\, c(p^-;p^+) > 0$:*

$$\int_{-\pi}^{\pi} |f\,g|\, dt \le c\left(p^-;p^+\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}, \ \forall f \in L_{p(\cdot)}(-\pi,\pi), \ \forall g \in L_{p'(\cdot)}(-\pi,\pi).$$

The following theorem is true.

Theorem 3. *[24] Let $p(\cdot) \in WL(-\pi,\pi)$: $p^- > 1$. Then the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for $L_{p(\cdot)}(-\pi,\pi)$.*

Along with the system E_λ, consider its particular case

$$E_\lambda^\alpha = \left\{e^{i(n+\alpha\, sign\, n)\,t}\right\}_{n\in Z},$$

where $\alpha \in C$ is some parameter. In [8], the following theorem was proved.

Theorem 4. *[8] Let $p(\cdot) \in WL(-\pi,\pi)$: $p^- > 1$. If the following inequalities satisfies*

$$-\frac{1}{p'(\pi)} < 2Re\,\alpha < \frac{1}{p(\pi)},$$

then the system E_λ^α forms a basis for $L_{p(\cdot)}(-\pi,\pi)$.

Using the method of proving this theorem, completely analogous to [1], the validity of the following theorem is established.

Theorem 5. *Let $p(\cdot) \in WL(-\pi,\pi)$: $p^- > 1$. System E_λ^α forms a basis for $L_{p(\cdot)}(-\pi,\pi)$ if and only if it is isomorphic in it to the classical system of exponents $E_\lambda^0 = \left\{e^{int}\right\}_{n\in Z}$.*

In obtaining the main results, we will essentially use the following $L_{p(\cdot)}$-analogue of the Theorem 2.2 [25].

Statement 2. *Let $p(\cdot) \in WL(-\pi, \pi)$: $p^- > 1$. System E_λ is complete in $L_{p(\cdot)}(-\pi, \pi)$ if and only if $\overline{L[E_\lambda]}$ contains an exponent $e^{i\lambda t}$ different from E_λ.*

Proof. The necessary is obvious. Let $e^{i\lambda t} \notin E_\lambda$ and $e^{i\lambda t} \notin \overline{L[E_\lambda]}$. The definition of the norm directly implies the following relation

$$\|f\,g\|_{p(\cdot)} \le \|f\|_{L_\infty(-\pi,\pi)}\,\|g\|_{p(\cdot)}\,.$$

This inequality immediately implies that $1 \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]}$. It is quite obvious that if $|f(t)| \le |g(t)|$, a.e. $t \in (-\pi, \pi)$, then $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$. Consequently

$$\left\|\int_0^x \left(1-\sum_n a_n e^{i(\lambda_n-\lambda)\,t}\right) dt\right\|_{p(\cdot)} = \left\|x-\sum_n b_n e^{i(\lambda_n-\lambda)\,x}+\sum_n b_n\right\|_{p(\cdot)},$$

where $b_n = \frac{a_n}{i(\lambda_n-\lambda)}$. It is clear that $\sum_n b_n \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]}$. On the other hand, we have

$$\left\|\int_0^x \left(1-\sum_n a_n e^{i(\lambda_n-\lambda)\,t}\right) dt\right\|_{p(\cdot)} \le$$

$$\le c\left\|\int_{-\pi}^{\pi}\left|e^{i\lambda t}-\sum_n a_n e^{i\lambda_n t}\right| dt\right\|_{p(\cdot)} \le c\left\|e^{i\lambda t}-\sum_n a_n e^{i\lambda_n t}\right\|_{p(\cdot)}.$$

It follows from these relations that $x \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]}$. Continuing this process, as a result, we obtain $L\left[\{x^n\}_{n\in Z_+}\right] \subset \overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]}$. Since the polynomials are dense in $C[-\pi,\pi]$, then it follows that $C[-\pi,\pi] \subset \overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]}$. Since $C[-\pi,\pi]$ (by Property 2.1 [12]) is dense in $L_{p(\cdot)}(-\pi,\pi)$, then it follows that $\overline{L\left[\left\{e^{i(\lambda_n-\lambda)\,t}\right\}_{n\in Z}\right]} = L_{p(\cdot)}(-\pi,\pi) \Rightarrow \overline{L\left[\left\{e^{i\lambda_n t}\right\}_{n\in Z}\right]} = L_{p(\cdot)}(-\pi,\pi)$.

The statement is proved. ◀

3. Main Results

Consider the following system of exponents

$$E_\lambda \equiv 1 \bigcup \left\{e^{\pm i\lambda_n t}\right\}_{n\in N},$$

where $\lambda_n = \sqrt[m]{|P_m(n)|}$,$P_m(n) = n^m + \alpha_{m-1}n^{m-1} + ... + \alpha_0$ is a polynomial of degree m, $m \in N$. Let us find the asymptotics λ_n as $n \to \infty$. Let $f(x) =$

$(1+x)^{\frac{1}{m}}$, $|x| < 1$. We have $f^{(n)}(x) = \frac{1}{m}\left(\frac{1}{m}-1\right)...\left(\frac{1}{m}-n+1\right)(1+x)^{\frac{1}{m}-n} \Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)...\left(\frac{1}{m}-n+1\right)}{n!} \Rightarrow$

$$\Rightarrow \sup_n \left|\frac{f^{(n)}(0)}{n!}\right| \leq 1.$$

From these relations it immediately follows

$$f(x) = 1 + f'(0)\,x + \underline{\underline{O}}\left(x^2\right) = 1 + \frac{1}{m}x + \underline{\underline{O}}\left(x^2\right),\ x \to 0. \tag{1}$$

Consequently (it is clear that for large n $P_m(n) > 0$)

$\lambda_n = \left(n^m + \alpha_{m-1}n^{m-1} + ... + \alpha_0\right)^{\frac{1}{m}} = n\left(1 + \frac{\alpha_{m-1}}{n} + \underline{\underline{O}}\left(\frac{1}{n^2}\right)\right)^{\frac{1}{m}} =$ / by formula (1)/

$$= n\left(1 + \frac{\alpha_{m-1}}{m}\frac{1}{n} + \underline{\underline{O}}\left(\frac{1}{n^2}\right)\right) = \left(n + \frac{\alpha_{m-1}}{m} + \underline{\underline{O}}\left(\frac{1}{n}\right)\right),\ n \to \infty.$$

Assume $\mu_n = n + \frac{\alpha_{m-1}}{m}$, $n \in N$. Considering the obvious inequality

$$\left|e^{ix} - e^{iy}\right| \leq 2\,|x-y|\ ,\ \forall x, y \in R,$$

we have

$$\left|e^{i\lambda_n t} - e^{i\mu_n t}\right| \leq 2\pi\,|\lambda_n - \mu_n| = \underline{\underline{O}}\left(\frac{1}{n}\right),\ n \to \infty.$$

This estimate immediately implies

Lemma. *System E_λ is r-close in $L_{p(\cdot)}(-\pi,\pi)$ for $p^- \geq 1$ to the system of exponents*

$$E_\mu = 1\bigcup\left\{e^{\pm i\mu_n t}\right\}_{n\in N},$$

for $\forall r > 1$.

Indeed, we have

$$\sum_n \left\|e^{i\lambda_n t} - e^{i\mu_n t}\right\|_{p(\cdot)}^r \leq c\sum_n \frac{1}{n^r} < +\infty.$$

Suppose that the condition

$$-\frac{1}{2p'(\pi)} < \frac{\alpha_{m-1}}{m} < -\frac{1}{2p(\pi)}$$

is satisfied. Let $p \in WL(-\pi,\pi) : p^- > 1$. Then it follows from Theorem 2.4 [8] that the system E_μ forms a basis for $L_{p(\cdot)}(-\pi,\pi)$. By Theorem 5, it is isomorphic to the system E_λ^0 in $L_{p(\cdot)}(-\pi,\pi)$. Suppose $e_n(t) = e^{int}$, $n \in Z$, and consider the following functionals:

$$e_n^*(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}dt,\ n \in Z.$$

Paying attention to Statement 1, we have

$$|e_n^*(f)| \leq c\|f\|_{p(\cdot)}\ , \forall f \in L_{p(\cdot)}(-\pi,\pi)\,,$$

where $c > 0$ is a constant depending only on $p(\cdot)$. Hence it follows that $\gamma = \sup_n \|e_n^*\| < +\infty$. It follows from the basicity of the system E_λ^0 in $L_{p(\cdot)}(-\pi,\pi)$ that the following relation

$$1 \leq \|e_n\|_{p(\cdot)}\ \|e_n^*\| \leq const < +\infty, \forall n \in Z,$$

holds. As $\|e_n\|_{p(\cdot)}\ = const \neq 0$, $\forall n \in Z$, then it follows from the previous relation that $\exists\, \delta > 0$:

$$0 < \delta \leq \|e_n^*\| \leq \gamma < +\infty, \forall n \in Z.$$

Let us show that $\exists\, r \in (1,2]$, with respect to which the system E_μ is r'-Besselian in $L_{p(\cdot)}(-\pi,\pi)$. First, we establish the validity of this fact with respect to the system E_λ^0. Indeed, let $r = \min\{p^-;2\}$. It is easy to see that the continuous embedding $L_{p(\cdot)}(-\pi,\pi) \subset L_r(-\pi,\pi)$, holds, i.e. $\exists\, c > 0$:

$$\|f\|_{L_r(-\pi,\pi)} \leq c\|f\|_{p(\cdot)}\ ,\ \forall f \in L_{p(\cdot)}(-\pi,\pi)\,. \tag{2}$$

It follows from the classical Hausdorff-Young theorem that $\exists\, c_r > 0$:

$$\left(\sum_n |e_n^*(f)|^{r'}\right)^{\frac{1}{r'}} \leq c_r\,\|f\|_{L_r(-\pi,\pi)}\,.$$

Taking into account (2), hence we obtain

$$\left(\sum_n |e_n^*(f)|^{r'}\right)^{\frac{1}{r'}} \leq c_r\,\|f\|_{p(\cdot)}\,, \forall f \in L_{p(\cdot)}(-\pi,\pi)\,,$$

and as a result the system E_λ^0 is a r'-Besselian in $L_{p(\cdot)}(-\pi,\pi)$. Let us show that the system E_μ is also a r'-Besselian in $L_{p(\cdot)}(-\pi,\pi)$. Indeed, let $T \in \left[L_{p(\cdot)}(-\pi,\pi)\right]$ be an automorphism such that $T[E_\mu] = E_\lambda^0$, i.e. T transfers system E_μ to system E_λ^0. From the relation

$$\delta_{nk} = e_n^*(e_k) = e_n^*(T[e_{\alpha;k}]) = (T^* e_n^*)(e_{\alpha;k})\ ,\ \forall n,\ k \in Z,$$

where

$$e_{\alpha;0} = 1,\ e_{\alpha;n} = e^{i\left(n+\frac{\alpha_{m-1}}{m} sign\, n\right)t}\ , \forall n \neq 0,$$

it follows that

$$T^* e_n^* = e_{\alpha;n}^*, \forall n \in Z,$$

where $\left\{e_{\alpha;n}^*\right\}_{n\in Z} \subset L_{p'(\cdot)}(-\pi,\pi)$ is a system biorthogonal to E_μ. Consequently

$$\left(\sum_n \left|e_{\alpha;n}^*(f)\right|^{r'}\right)^{\frac{1}{r'}} = \left(\sum_n |T^* e_n^*(f)|^{r'}\right)^{\frac{1}{r'}} = \left(\sum_n |e_n^*(Tf)|^{r'}\right)^{\frac{1}{r'}} \leq$$

$$\leq c\left\|Tf\right\|_{p(\cdot)} \leq c\left\|T\right\|_{\left[L_{p(\cdot)}(-\pi,\pi)\right]}\left\|f\right\|_{p(\cdot)}, \; \forall f \in L_{p(\cdot)}\left(-\pi,\pi\right).$$

This established that the system E_μ is r'-Besselian in $L_{p(\cdot)}(-\pi,\pi)$. Thus, the systems E_λ and E_μ satisfy all the conditions of Theorem 2.1 [6] and, as a result, the following theorem holds.

Theorem 6. *Let $p \in WL(-\pi,\pi) : p^- > 1$, and the following inequalities*

$$-\frac{1}{2p'(\pi)} < \frac{\alpha_{m-1}}{m} < \frac{1}{2p(\pi)}$$

holds. Then the following properties of the system E_λ are equivalent in $L_{p'(\cdot)}(-\pi,\pi)$: i) E_λ is complete; ii) E_λ is minimal; iii) E_λ ω-linearly independent; iv) E_λ forms a basis isomorphic to E_μ.

In what follows, we will assume that the conditions

$$\lambda_n \neq 0\,, \forall n \neq 0 \;\; \& \;\; \lambda_i \neq \lambda_j\,,\; i \neq j,$$

are fulfilled. Let all conditions of Theorem 6 be satisfied. Then it follows from the proof of Theorem 2.1 [6] that the system E_λ forms a defect basis for $L_{p(\cdot)}(-\pi,\pi)$ (i.e. after adding a finite number of elements to it and eliminating from it a finite number of elements, it forms a basis), and there exists a Fredholm operator $F \in \left[L_{p(\cdot)}(-\pi,\pi)\right]$, such that $F[E_\lambda] = E_\mu$, i.e. F transfers the system E_λ to the system E_μ. Consequently, it is quite obvious that for large $n_0 \in N$, the system $E_{\lambda;n_0} \equiv \left\{e^{\pm i\lambda_n t}\right\}_{|n|>n_0}$ is minimal in $L_{p(\cdot)}(-\pi,\pi)$, and its defect is equal to $2n_0+1$. Then it follows from Statement 2 that $e^{i\lambda_k t} \notin \overline{L[E_{\lambda;n_0}]}$ (the closure is taken in $L_{p(\cdot)}(-\pi,\pi)$, where $k : |k| \leq n_0$ is an arbitrary integer. As a result, we obtain that the system $\left\{e^{i\lambda_k t}\right\} \bigcup E_{\lambda;n_0}$ is minimal in $L_{p(\cdot)}(-\pi,\pi)$. Continuing this process, we finally obtain that the system $\left\{e^{i\lambda_k t}\right\}_{k:|k|\leq n_0} \bigcup E_{\lambda;n_0}$ is minimal in $L_{p(\cdot)}(-\pi,\pi)$, and as a result, as follows from Theorem 6, it forms a basis isomorphic to E_μ in $L_{p(\cdot)}(-\pi,\pi)$. Thus, the following final theorem is true.

Theorem 7. *Let $p \in WL(-\pi,\pi) : p^- > 1$, and the following conditions are fulfilled*

$$-\frac{1}{2p'(\pi)} < \frac{\alpha_{m-1}}{m} < \frac{1}{2p(\pi)}, \quad \lambda_n \neq 0\,, \quad \forall n \neq 0 \quad \& \quad \lambda_i \neq \lambda_j\,, \;\; i \neq j.$$

Then the system E_λ forms a basis isomorphic to the classical system of exponents E^0_λ in $L_{p(\cdot)}(-\pi,\pi)$.

In fact, under the conditions of the theorem, the system E_λ is isomorphic to the basis E_μ. Since the system E_μ is isomorphic to E^0_λ (by Theorem 5), it is clear that the system E_λ is also isomorphic to $E(x)$.

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