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SYNTHESIS OF ZONAL CONTROLS USING INFORMATION ON THE OBJECT'S STATE AT THE CURRENT AND PREVIOUS MOMENTS OF TIME

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Abstract. *It the work, it is assumed that only a part of the components of the phase state vector is controllable. To form the current values of the control actions, we use the measured values of the components both at the current and in some previous moments of time. As a result, the process under study is described by differential equations with time-lagging arguments in the phase variable. Another feature of the approach to the feedback control is that the parameters of the dependence of the control actions on the measured values of the state are constant on subsets (zones) of the phase space, into which it is divided in advance. We call such feedback parameters zonal. Due to the fact that the phase space is divided into a finite number of zones, the number of optimizable feedback parameters in the control problem is also finite. Accordingly, the original feedback control problem is reduced to a finite-dimensional optimization problem with the values of the zonal feedback parameters as design variables.*

Keywords: zone control, feedback, time lag, feedback parameters

Mathematics Subject Classification (2020): 49N35, 49M05, 49M41

1. Introduction

We consider the problem of optimal control of a dynamic process with continuous state feedback described by a system of ordinary differential equations. The issues of control of technical objects and technological processes with feedback within the framework of automatic control systems attracted attention of many scientists (mainly engineers) as early as in the 19th century. Starting from the 50s of the last century, the feedback optimal control theory began to actively develop, first, for objects with lumped parameters [3], [6], [7], and later, for objects with distributed parameters [2], [3], [6]. Various approaches have been developed for both state and output feedback control. In the works [2], [3],

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[5], they conduct the analysis of the state of research both in the optimal control and feedback control theories.

In our work, we assume that only a part of the components of the phase state vector is controllable. Such a situation may arise when there is no possibility of real-time measurement of some state parameters, or in the case when these measurements have large errors. These components include, for example, the rate of change in the phase state, usually measured using indirect methods or obtained by means of calculations. We propose to compensate for the inability to measure some of the components of the phase vector by using the measured values of the components at some previous points in time to form the values of control actions. As a result of using the dependence of the control actions on the state at some previous moments of time, the process under study will be described by differential equations with time-lagging arguments in the phase variable.

Another feature of the considered feedback problem is that the parameters of the dependence of the control actions on the measured values of the state are constant on subsets (zones) of the phase space, into which it is divided in advance. We call such feedback parameters "zonal". Because the phase space is divided into a finite number of zones, the number of optimizable feedback parameters in the control problem is also finite. Accordingly, the original feedback control problem is reduced to a finite-dimensional optimization problem, which consists of optimizing the values of the zonal feedback parameters. As a result, we have a specific instance of a parametric optimal control problem. We investigate the differentiability of the objective functional of the reduced problem with respect to the zonal values of the feedback parameters, and obtain formulas for the components of the gradient of the objective functional. They make it possible to formulate necessary optimality conditions of the synthesized parameters, as well as to use them for carrying out computer experiments to solve some model test problems using first-order numerical optimization methods.

2. Problem Statement

We consider the control problem for a dynamic object described by a nonlinear system of ordinary differential equations

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad t \in (0, T], \qquad x \in \mathbb{R}^n. \tag{1}$$

Here the $n-$ dimensional vector function $x(\cdot)$ is a phase variable; $u(\cdot) \in U \subset \mathbb{R}^r$ is a piecewise continuous $r-$dimensional control vector-function belonging to a convex compact set of feasible values U; the function $f(\cdot, \cdot)$ is continuously differentiable in its arguments. A set X^0 of possible initial states of the object is given

$$x(t) = x^0 = const, \quad t \leq 0, \quad x^0 \in X^0 \subset \mathbb{R}^n, \tag{2}$$

and also given is the corresponding distribution function $\rho_{X^0}\left(x^0\right)$ of values x^0 on the set X^0 such that

$$0 \leq \rho_{X^0}\left(x^0\right), \quad \int_{X^0} \rho_{X^0}\left(x^0\right) dx^0 = 1.$$

The objective functional is specified as

$$F(u) = \int_{X^0} \left\{ \int_0^T f^0(x,u)dt + \Phi\left(x(T)\right) \right\} \rho_{X^0}\left(x^0\right)\ dx^0 \tag{3}$$

defining the quality of the control function $u(\cdot)$ of the object on average for all possible initial states $x^0 \in X^0$.

Suppose that during the object's operation it is possible to continuously measure in time the state of the part $\widetilde{x}(t) \in \mathbb{R}^m$ of the components of the phase vector $x(t)$, with $m \leq n$. Without loss of generality, we will further assume that the vector function $\widetilde{x}(t)$ consists of the first m components of the phase vector $x(t)$. To form the current control action at time t, in addition to the state value $\widetilde{x}(t)$, we also use the previously measured values $\widetilde{x}(t-\tau)$. The choice of the lag time τ is made depending on the rate of change in the object's state, namely, at high speeds, the time τ is to be chosen small. The choice of the lag time τ also depends on the accuracy of measurements; with low accuracy, τ is chosen large enough so that the dynamics of the change in the object's state would exceed the accuracy of measurements.

To form the current values of the control actions using feedback, we introduce the concept of "zonal control". Let $\Omega \subseteq \mathbb{R}^m$ be the set of all possible states of the components $\widetilde{x}(t)$ of the object's state $x(\cdot)$, which they can take under all possible values of the initial state $x^0 \in X^0$ and control $u(t) \in U$. We split the set Ω into L subsets (zones) $\Omega_i \subset \Omega$, $i = 1, 2, \ldots, L$, such that

$$\bigcup_{i=1}^{L} \Omega_i = \Omega, \quad \operatorname{int}(\Omega_i) \cap \operatorname{int}(\Omega_j) = \varnothing, \qquad i, j = 1, 2, \ldots, L, \quad i \neq j.$$

The current value of the control at the point of time t will be built in the form of the following linear dependence on the values of the components of the phase variable $\widetilde{x}(t)$ measured at times t and $t - \tau$:

$$u(t) = K_1^i \widetilde{x}(t) + K_2^i \widetilde{x}(t-\tau), \quad \widetilde{x}(t) \in \Omega_i. \tag{4}$$

Let's introduce the notation

$$K^i = \left(K_1^i, K_2^i\right), \quad K = \left(K^1, K^2, \ldots, K^L\right).$$

The dependence (4) in component-wise form can be written as follows:

$$u_s(t) = \sum_{j=1}^{m} \left[k_{1s,j}^i \widetilde{x}_j(t) + k_{2s,j}^i \widetilde{x}_j(t-\tau)\right], \ \widetilde{x}(t) \in \Omega_i, \quad s = 1, 2, \ldots, r. \tag{5}$$

Here $K_1^i = \left(k_{1s,j}^i\right)$ and $K_2^i = \left(k_{2s,j}^i\right)$ are constant matrices of dimension $r \times m$, determining the optimizable feedback parameters when the components $\widetilde{x}(t)$ of the phase variable belong to the i^{th} subset (zone) Ω_i. Controls of the form (4) will be called "zonal" [1], [4]. For the sake of simplicity, it is assumed that in (4) τ is sufficiently small and for $\widetilde{x}(t) \in \Omega_i$, $\widetilde{x}(t-\tau) \in \Omega_i$.

To build the subsets Ω_i constructively, we proceed as follows. We will assume that each of the components $\widetilde{x}_s(t)$ of the phase vector $\widetilde{x}(t)$ under all possible initial conditions $x^0 \in X^0$ and controls $u(t) \in U$ belongs to some given finite interval:

$$\underline{\omega}_s \leq \widetilde{x}_s(t) \leq \overline{\omega}_s, \quad t \in [0,T], \quad s = 1,2,\ldots,m. \tag{6}$$

Let's divide each of the intervals $[\underline{\omega}_s, \overline{\omega}_s]$ into l_s subintervals by the points

$$\omega_s^j, \quad j = 0,1,2,\ldots,l_s; \quad \omega_s^0 = \underline{\omega}_s, \quad \omega_s^{l_s} = \overline{\omega}_s.$$

Let's introduce the notation for the $m-$dimensional parallelepipeds:

$$\Omega_i = \left\{x(t) \in \mathbb{R}^m : \omega_s^{i_s} \leq x_s(t) \leq \omega_s^{i_s+1}, s = 1,2,\ldots,m\right\},$$

$$i_s = 0,1,2,\ldots,l_s - 1; \quad s = 1,2,\ldots,m. \tag{7}$$

And the notation for the set of all vertices of the i^{th} zone (parallelepiped)

$$W_i = \left\{w = (w_1, w_2, \ldots, w_m) : w_j = \omega_j^{i_j} \wedge \omega_j^{i_j+1}, \quad j = 1,2,\ldots,m\right\}.$$

Here and below, as will be clear from the context, $i = (i_1, i_2, \ldots, i_m)$ designates an $m-$dimensional multi-index, i.e. the $m-$dimensional number of the corresponding zone Ω_i. It is clear that

$$\Omega = \bigcup_{i_1=0}^{l_1} \ldots \bigcup_{i_m=0}^{l_m-1} \Omega_i$$

and the total number of zonal areas is

$$L = \prod_{i=1}^{m} l_i.$$

Substituting the feedback control into (4) using the values of the observable components $\widetilde{x}(t)$ at the current and previous points of time into the system of equations (1), we obtain:

$$\dot{x}(t) = f\left(x(t), K_1^i \widetilde{x}(t) + K_2^i \widetilde{x}(t-\tau)\right), \quad t \in (0,T], \quad \widetilde{x}(t) \in \Omega_i. \tag{8}$$

Here and below, $x(t) = x\left(t; x^0, K\right)$ is a piecewise continuously differentiable vector-function that is a solution to system (8) for given admissible initial conditions $x^0 \in X^0$ and parameters of the feedback K. The system of differential equations (8) is a system with a constant lag argument in the phase variable. Under the prescribed conditions on the vector-function $f\left(x(\cdot), u(\cdot)\right)$, The solution to the Cauchy problem, $x(t) = x\left(t;\ x^0,\ K\right)$, for arbitrarily given constant feedback parameters $K = (K_1,\ K_2)$ and the initial condition $x^0 \in X^0$ has a solution from the class of continuous piecewise differentiable functions.

Let $T_{i,\ x^0, K} \subset [0,T]$ denote the set of points of time t at which the components $\widetilde{x}(t)$ of the phase trajectory $x(t)$ with the initial point x^0 and feedback parameters $K^i =$

$\left(K_1^i, K_2^i\right)$, $K = \left(K^1, K^2, \ldots, K^L\right)$, belong to the set (zone) Ω_i. The sets $T_{i,\ x^0,K}$ may be multiply connected, i.e.

$$T_{i,\ x^0,K} = \sum_{j=1}^{l_i} T_{i,\ x^0,K}^j \ .$$

This means that the components $\widetilde{x}(t)$ of the phase trajectory $x(t)$ in l_i separate time intervals belong to the set Ω_i:

$$\widetilde{x}(t) \in \Omega_i, \quad t \in T_{i,\ x^0,K}^j, \quad j = 1, 2, \ldots, l_i.$$

It is clear that $T_{i,\ x^0,K} = \varnothing$ means that the components $\widetilde{x}\left(t; x^0, K\right)$ of the phase trajectory for $t \in [0,\ T]$ did not intersect the domain Ω_i. Note that in the case when the admissible control region U has a simple structure, for example, it is an $r-$dimensional parallelepiped

$$U = \left\{u \in \mathbb{R}^r : \ \underline{u}_s \le u_s(t) \le \overline{u}_s, s = 1, 2, \ldots, r\right\},$$

where $\underline{u}_s$ and $\overline{u}_s$ are given values, the constraints on the corresponding values of the zonal feedback parameters can also be reduced to a simple form. Taking into account (4)-(6), the following linear constraints on the feedback parameters will take place:

$$\underline{u} \le \sum_{j=1}^{m} \left(k_{1s,j}^i + k_{2s,j}^i\right) w \le \overline{u}, \quad \forall w \in W_i, \quad i = 1, 2, \ldots, L, \tag{9}$$

i.e. conditions (9) must be satisfied at all vertices of all parallelepipeds Ω_i, $i = (i_1, \ldots, i_m)$, $0 \le i_j \le l_j - 1$, $j = 1, 2, \ldots, L$. It is possible that feedback with the object can be carried out only at given discrete points of time $t_s \in [0, T]$, $s = 0, 1, 2, \ldots, N_t$. Then we choose the current value of the control in the form:

$$u(t) = K_1^i x\left(t_s\right) + K_2^i x\left(t_s - \tau\right), \quad t \in \left[t_s, t_{s+1}\right), \quad x\left(t_s\right) \in \Omega_i. \tag{10}$$

As can be seen from (10), the values of the feedback parameters over the time interval $[t_s, t_{s+1})$ are determined by the values of K^i of the zone Ω_i, to which the controllable components of the state $\widetilde{x}(t)$ belonged at the moment of measuring, t_s.

Let's introduce the notation:

$$S_{x^0,K}^i = \left\{s : \widetilde{x}\left(t; x^0, K\right) \in \Omega_i, \ s = 0, 1, 2, \ldots, N_t\right\}, \quad i = 1, 2, \ldots, L;$$

$$\mathcal{T}_{i,x^0,K} = \bigcup_{s \in S_{x^0,K}^i} \left[t_s, t_{s+1}\right).$$

Substituting controls in the form of feedback (10) into (1), we will have:

$$\dot{x}(t) = f\left(x(t), K_1^i x\left(t_s\right) + K_2^i x\left(t_s - \tau\right)\ \right),$$

when $t \in [t_s, t_{s+1})$ and $x(t_s) \in \Omega_i$ or $t \in \mathcal{T}_{i,x^0,K}$. Constraints (9) on the zonal feedback parameters will not change in this case.

Thus, the original optimal control problem (1)-(3), taking into account the feedback with a part of the components of the phase variable in the form (4) and (5), has been reduced to the following optimization problem: Find the feedback parameters K_1^i and K_2^i, $i = 1,\ 2, \ldots,\ L$, – $(r\ \times\ m)$–dimensional constant matrices satisfying linear constraints (9), for which the corresponding solutions of the Cauchy problems with respect to the system of differential equations (8) involving the lag argument with the set of initial conditions (2) minimizes the given functional (3), which already depends only on K, i.e. $F(K)$. The dimension of the obtained parametric optimal control problem is $M = r \times m \times L$, i.e. $K \in \mathbb{R}^M$. Note that even if the original control problem (1)-(3) is convex, then the feedback control problem (2), (3), (8), and (9), despite the linear dependence (4), in the general case may be nonconvex with respect to K.

Let us make the following remark in connection with the choice of the dependence of the control action on the values of a part of the state components in the form (4). If it is possible to control all components of the state vector $x(t)$, it is known that the feedback, for example, in the case of linear systems of the second order

$$\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t) \tag{11}$$

is selected in the following form:

$$u(t) = \ \alpha_1(t)x(t) + \alpha_2(t)\dot{x}(t),$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are feedback parameters. Assuming that the components of the velocity state $\dot{x}(t)$ are not directly controllable, we replace the dependence (11) by the one "close" to it:

$$u(t) = \ \alpha_1(t)x(t) + \frac{\alpha_2(t)\left[x(t) - x(t-\tau)\right]}{\tau}.$$

Further, assuming that the feedback parameters do not depend on time, but on the number of the subdomain to which the current state $x(t)$ belongs, we obtain the feedback in the form proposed in (4).

3. Derivation of the Formulas

To investigate the differentiability of the functional of problem (2), (3), (8), and (9), and obtain formulas for the gradient of the objective functional (3) with respect to the feedback parameters, we use the well-known technique of incrementing the optimizable parameters. First of all, note that the mutual independence of the initial conditions $x^0 \in X^0$ for the gradient of functional (3) implies the following equality:

$$\nabla_K\ F(K) = \nabla \int_{X^0} \left\{ \int_0^T f^0\left(x(t),\ K\right) dt + \Phi\left(x(T)\right) \right\} \rho_{X^0}\left(x^0\right) dx^0 =$$

$$= \int_{X^0} \nabla \left\{ \int_0^T f^0\left(x(t),\ K\right) dt + \Phi\left(x(T)\right) \right\} \rho_{X^0}\left(x^0\right) dx^0 =$$

$$= \int_{X^0} \nabla\ I\left(K; x_0\right) \rho_{X^0}\left(x^0\right) dx^0. \tag{12}$$

Here we have introduced the notation for the functional

$$I\left(K;x^0\right)=\int_0^T f^0\left(x(t),K\right)dt+\Phi\left(x(T)\right)$$

and instead of the control $u(t)$, the feedback parameters K are used, which determine the control $u(t)$. Therefore, we will study the functional $I\left(K;x^0\right)$ for an arbitrary given admissible initial condition $x^0\in X^0$.

Let $K=(K_1,K_2)$ be the zonal parameters of the feedback, which correspond, according to (4) and (5), to the control $u(t)=u(t;K)$ and the solution $x(t)=x\left(t;K,\ x^0\right)$ of the Cauchy problem (2) and (8). Suppose that the parameters K obtained an increment $\Delta K=(\Delta K_1,\Delta K_2)$: $K_1^{\Delta}=K_1+\Delta K_1$, $K_2^{\Delta}=K_2+\Delta K_2$. The corresponding increments will receive the control $u(t)$:

$$u^{\Delta}\left(t;K^{\Delta}\right)=u(t;K)+\mathrm{u}(t;K),\quad t\in[0,T],$$

and the phase vector $x(t)$:

$$x^{\Delta}\left(t;K^{\Delta},x^0\right)=x\left(t;K,x^0\right)+\mathrm{x}\left(t;K,x^0\right),\quad t\in[0,T],$$

including the controlled components $\widetilde{x}(t)$ of the phase variable:

$$\widetilde{x}^{\Delta}\left(t;K^{\Delta},x^0\right)=\widetilde{x}\left(t;K,x^0\right)+\Delta\widetilde{x}\left(t;K,x^0\right),\quad t\in[0,T].$$

According to (2), the following conditions hold:

$$\Delta x(t)=0,\quad \Delta\widetilde{x}(t)=0,\quad t\le 0.$$

It is clear that for $\widetilde{x}(t)\in\Omega_i$

$$\Delta u(t;K)=\Delta K_1^i\widetilde{x}_1(t)+\Delta K_2^i\widetilde{x}_2(t-\tau)+$$
$$+K_1^i\Delta\widetilde{x}(t)+K_2^i\Delta\widetilde{x}(t-\tau)+\Delta K_1^i\Delta\widetilde{x}(t)+\Delta K_2^i\widetilde{x}(t-\tau). \tag{13}$$

The system of differential equations for the increment of the phase variable up to terms of the first order of accuracy is easily determined from (8) and (13):

$$\Delta\dot{x}(t)=\frac{\partial f(t)}{\partial x}\,\Delta x(t)+\frac{\partial f(t)}{\partial u}\times$$
$$\times\left[\Delta K_1^i\widetilde{x}(t)+K_1^i\Delta\widetilde{x}(t)+\Delta K_2^i\widetilde{x}(t-\tau)+K_2^i\Delta\widetilde{x}(t-\tau)\right]$$

when $t\in T_{i,x^0,K}$. Using the well-known Grönwall's lemma and the "steps" method for studying Cauchy problems with respect to differential equations involving a lagging argument, one can prove the validity of the estimate:

$$\|\Delta x(t)\|_{L_2^n\ [0,\ T]}\le C\,\|K\|_{\mathbb{R}^{\mathrm{M}}}\ .$$

Hereinafter, for brevity, the following notation is used:

$$f(t)=f\left(x(t),u(t)\right),\quad f(t\pm\tau)=f\left(x(t\pm\tau),u(t\pm\tau)\right),$$

$$f^0(t) = f^0\left(x(t), u(t)\right),$$
$$f^0(t \pm \tau) = f^0\left(x(t \pm \tau), u(t \pm \tau)\right).$$

Let us consider an estimate for the increment of the functional $I\left(K; x^0\right)$:

$$\Delta I\left(K; x^0\right) = I\left(K + \Delta K; x^0\right) - I\left(K; x^0\right) =$$
$$= \int_0^T \left[\frac{\partial f^0(t)}{\partial x}\ \Delta x(t) + \frac{\partial f^0(t)}{\partial u}\ \Delta u(t)\right] dt + \mathcal{R}.$$

Here $\mathcal{R} = \mathcal{R}\left(\|K\|, \|\Delta x\|\right)$ is the remainder term of the second order of smallness with respect to $\|K\|$ and $\|\Delta x\|$, where $\|.\|$ are any equivalent norms of the corresponding spaces. Let's move the right-hand sides of equations (9) to the left, scalar multiply the resulting vector expression, which is equal to zero, by an arbitrary $n-$dimensional vector function $\psi(t)$, and add to the integrand (12):

$$\Delta I\left(K; x^0\right) = \int_0^T \left\{\frac{\partial f^0(t)}{\partial x}\ \Delta x(t) + \frac{\partial f^0(t)}{\partial u}\ \Delta u(t) + \right.$$
$$\left. + \psi^*(t)\left[\ \Delta \dot{x}(t) - \frac{\partial f(t)}{\partial x}\Delta x(t) - \frac{\partial f(t)}{\partial u}\Delta u(t)\right]\right\} dt + \mathcal{R}.$$

Here $*$ is the transposition sign.

If we use integration by parts, after grouping we will have:

$$\Delta I\left(K; x^0\right) = \int_0^T \left[-\dot{\psi}^*(t) - \psi^*(t)\frac{\partial f(t)}{\partial x} - \frac{\partial f(t)}{\partial x}\right] \Delta x(t) dt +$$
$$+ \int_0^T \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f(t)}{\partial u}\right] \Delta u(t) dt + \mathcal{R}. \tag{14}$$

We then transform the 2^{nd} integral in (14), taking into account (5), (7), and (8), sequentially. We then have:

$$\int_0^T \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f(t)}{\partial u}\right] \Delta u(t) dt =$$
$$= \sum_{i=1}^{L} \int_{T_{i,x^0,K}} \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f(t)}{\partial u}\right] \Delta u(t) dt =$$
$$= \sum_{i=1}^{L} \int_{T_{i,x^0,K}} \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f(t)}{\partial u}\right] \left[\Delta K_1^i \widetilde{x}(t) + K_1^i \Delta \widetilde{x}(t)\right] dt +$$
$$+ \sum_{i=1}^{L} \int_{T_{i,x^0,K}} \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f(t)}{\partial u}\right] \left[\Delta K_2^i \widetilde{x}(t-\tau) + K_2^i \Delta \widetilde{x}(t-\tau)\right] dt =$$
$$= S_1 + S_2. \tag{15}$$

In the second term, S_2, we change the time variable:

$$\xi = t - \tau,\ t = \xi + \tau,\ \xi \in [-\tau,\ T - \tau].$$

Since $\psi(t)$ is arbitrary, we will assume:

$$\psi(t) = 0,\ t \in [T - \tau,\ T]. \tag{16}$$

Then we will have for S_2:

$$\sum_{i=1}^{L} \int_{T_{i,\ x^0,K} \subset [-\tau,\ T-\tau]} \left[\frac{\partial f^0(\xi+\tau)}{\partial u} - \psi^*(\xi+\tau) \frac{\partial f(\xi+\tau)}{\partial u} \right] \times$$
$$\times \left[\Delta K_2^i \widetilde{x}(\xi) + K_2^i \Delta \widetilde{x}(\xi) \right] d\xi. \tag{17}$$

Renaming the variable ξ once again by t, taking into account (14), (15), (16), and (17), we will have:

$$\Delta I\left(K; x^0\right) = \sum_{i=1}^{L} \int_{T_{i,\ x^0,K}} \left[-\dot{\psi}^*(t) - \psi^*(t) \frac{\partial f(t)}{\partial x} - \frac{\partial f^0(t)}{\partial x} + \right.$$
$$\left. + \left(\frac{\partial f^0(t)}{\partial u} - \psi^*(t) \frac{\partial f(t)}{\partial u} \right) K_1^i \right] \Delta \widetilde{x}(t) dt +$$
$$+ \sum_{i=1}^{L} \int_{T_{i,\ x^0,K} \subset [-\tau,\ T-\tau]} \left[\frac{\partial f^0(t+\tau)}{\partial u} - \psi^*(t+\tau) \frac{\partial f(t+\tau)}{\partial u} \right] K_2^i \Delta \widetilde{x}(t) dt +$$
$$+ \sum_{i=1}^{L} \int_{T_{i,\ x^0,K}} \left[\frac{\partial f^0(t)}{\partial u} - \psi^*(t) \frac{\partial f(t)}{\partial u} \right] \Delta K_1^i \widetilde{x}(t) dt +$$
$$+ \sum_{i=1}^{L} \int_{T_{i,\ x^0,K} \subset [-\tau,\ T]} \left[\frac{\partial f^0(t+\tau)}{\partial u} - \psi^*(t+\tau) \frac{\partial f(t+\tau)}{\partial u} \right] \Delta K_2^i \widetilde{x}(t) dt + \mathcal{R}. \tag{18}$$

Since the function $\psi(t)$ is arbitrary, in addition to condition (16), we require that it be a solution to the following system of differential equations:

$$\dot{\psi}^*(t) = \begin{cases} -\psi^*(t) \frac{\partial f(t)}{\partial x} - \frac{\partial f^0(t)}{\partial x} + \left(\frac{\partial f^0(t)}{\partial x} - \psi^*(t) \frac{\partial f(t)}{\partial t} \right) K_1^i, \\ \quad \text{when } t \in T_{i,\ x^0,K} \subset [\ T - \tau,\ T], \\ -\psi^*(t) \frac{\partial f(t)}{\partial x} - \frac{\partial f^0(t)}{\partial x} + \left(\frac{\partial f^0(t)}{\partial x} - \psi^*(t) \frac{\partial f(t)}{\partial t} \right) K_1^i + \\ \quad + \left[\frac{\partial f^0(t+\tau)}{\partial x} - \psi^*(t+\tau) \frac{\partial f(t+\tau)}{\partial u} \right] K_2^i, \\ \quad \text{when } t \in T_{i,\ x^0,K} \subset [-\tau,\ T - \tau]. \end{cases} \tag{19}$$

The Cauchy problem (16) and (19) with initial data given at the right end will be called adjoint. Its solution $\psi(x,t)$ is a continuous and almost everywhere continuously differentiable function.

By virtue of the above estimate for the remainder of the functional increment:

$$\mathcal{R} \leq o\left(\|\Delta x(t)\|, \|\Delta K\|\right)$$

and the known estimate of the increment of the solution for the Cauchy problem:

$$\|\Delta x(t)\| \leq \alpha \|\Delta K\|, \quad \alpha \geq 0,$$

it follows that the functional $I\left(K; x^0\right)$ is differentiable with respect to the feedback parameters K for an arbitrary admissible initial condition $x^0 \in X^0$. Then the components of the gradient of the functional $I\left(K; x^0\right)$ for an arbitrary given admissible initial state $x^0 \in X^0$ with respect to the matrix parameters of the feedback $K^i = \left(K_1^i, K_2^i\right)$ from formula (18) are defined as the main parts with linear increments of the corresponding parameters:

$$\nabla_{K_1^i} I\left(K,\ x^0\right) = \int\limits_{T_{i,\ x^0,K}} \left(\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f^0(t)}{\partial u}\right)^* \widetilde{x}(t)dt,$$

$$\nabla_{K_2^i} I\left(K,\ x^0\right) = \int\limits_{T_{i,\ x^0,K} \subset [-\tau,\ T-\tau]} \left(\frac{\partial f^0(t+\tau)}{\partial u} - \psi^*(t+\tau)\frac{\partial f^0(t+\tau)}{\partial u}\right)^* \widetilde{x}(t)dt.$$

Here $x(t)$ and $\psi(t)$ are solutions of the direct (2) and (8), and the corresponding adjoint (16) and (19) Cauchy problems for the current values of the zonal feedback parameters $K^i = \left(K_1^i, K_2^i\right)$, $i = 1, 2, \ldots, L$, and a given admissible initial condition $x^0 \in X^0$. The vector function $\psi(t) = \psi(t; x^0, K)$ is a solution of the adjoint Cauchy problem (16) and (19) corresponding to the solution $x(t) = x(t; x^0, K)$ of the Cauchy problem (2) and (8) for admissible initial state x^0 and feedback parameters K. Thus, taking into account (12), the following theorem can be considered proven.

Theorem 1. *Under the accepted assumptions for the functions participating in the optimal control problem (1)-(3), the functional (3) is differentiable with respect to the parameters of linear feedback (4) with the part $\widetilde{x}(t)$ of the components of the phase vector $x(t)$, and the components of the functional gradient are determined by the following formulas:*

$$\nabla_{K_1^i} F(K) = \int\limits_{X^0} \int_{T_{i,x^0,K}} \left(\frac{\partial f^0(t)}{\partial u} - \psi^*(t)\frac{\partial f^0(t)}{\partial u}\right)^* \widetilde{x}(t)\rho_{X^0}\left(x^0\right) dt dx^0, \tag{20}$$

$$\nabla_{K_2^i} F(K) = \int\limits_{X^0} \int_{T_{i,\ x^0,K} \subset [-\tau,\ T-\tau]} \left(\frac{\partial f^0(t+\tau)}{\partial u} - \psi^*(t+\tau)\frac{\partial f^0(t+\tau)}{\partial u}\right)^* \times$$

$$\times \widetilde{x}(t)\rho_{X^0}\left(x^0\right) dt dx^0. \tag{21}$$

Next, we formulate the necessary conditions for optimality of the zonal values of the feedback parameters in the well-known variational form.

Theorem 2. *Let the functions $f(x,u)$, $f^0(x,u)$, and $\Phi(x)$ be continuously differentiable with respect to their arguments, $\widehat{K} = \left(\widehat{K}_1, \widehat{K}_2\right)$ be the optimal values of the zonal feedback parameters given in the form (4) that minimize functional (3) in problem (2), (3), and (8). Then, for all admissible values of K satisfying (9), the following inequality holds:*

$$\left\langle \nabla_K F(K), K - \widehat{K} \right\rangle \geq 0,$$

where $\nabla_K F(K)$ is the gradient of the functional (3) defined by formulas (19), (20), and (21).

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