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INVERSE PROBLEM FOR RECONSTRUCTING THE RIGHT SIDE OF A ONE-DIMENSIONAL WAVE EQUATION

I.K. GADIMOV, Kh.M. GAMZAEV*,
S.O. HUSEYNZADE, S.R. KARIMOVA

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Abstract. *An inverse problem of recovering the unknown parameter of an external source for a one-dimensional wave equation with a nonlocal additional condition is considered. It is assumed that the unknown source parameter depends only on time. By integration, the problem is transformed to an inverse boundary value problem with local conditions. A difference analogue of the differential problem in the form of an implicit difference scheme is constructed and a non-iterative computational algorithm for solving the resulting system of difference equations is proposed. As a result, an explicit formula is obtained for determining the approximate value of the sought parameter for each discrete value of the time variable.*

Keywords: wave equation, inverse problem, nonlocal condition, difference problem

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1. Introduction

It is known that in the mathematical modeling of many processes of heat and mass transfer, wave processes [2], [3], [12], problems of recovering unknown parameters of

* Corresponding author.

Ilgar K. Gadimov
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: ilqar89@mail.ru

Khanlar M. Gamzaev
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: xan.h@rambler.ru

Sevil O. Huseynzade
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: sevil.huseinzade@asoiu.edu.az

Sevinj R. Karimova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: sevinc.kerimova@asoiu.edu.az

external sources often arise. These problems belong to the class of inverse problems of mathematical physics associated with the restoration of the right-hand sides of partial differential equations. In such problems, in addition to solving differential equations, it is required to restore the unknown parameters of external sources. Inverse problems of recovering the right-hand sides of partial differential equations of parabolic type were studied in many works [1], [2], [5], [6], [9], [11]. There is also an extensive literature on inverse problems of reconstructing the right-hand sides of wave equations [4], [7], [8], [9]. However, most of these works mainly investigate the solvability of inverse problems for wave equations, the existence and uniqueness of their solutions.

The aim of this work is to develop a computational algorithm for the numerical solution of the inverse problem of reconstructing the right-hand side of a one-dimensional wave equation.

2. Statement Problem

Let us consider a one-dimensional wave equation with a source

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \nu(t)\frac{\partial^2 u(x,t)}{\partial x^2} + \lambda(t)u(x,t) + q(t)F(x,t), \quad 0 < x < l, \quad 0 < t \le T, \tag{1}$$

with the following conditions:

$$u(x,0) = \varphi_1(x), \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = \psi_1(x), \tag{3}$$

$$\frac{\partial u(0,t)}{\partial x} = \theta(t), \tag{4}$$

$$u(l,t) = p(t). \tag{5}$$

It is known that the direct problem for equation (1) consists in determining the function $u(x,t)$ from equation (1) with given coefficients $\nu(t) > 0$, $\lambda(t)$, right-hand side $q(t)F(x,t)$ and conditions (2)-(5).

Suppose that, in addition to the function $u(x,t)$, the unknown is also the function $q(t)$ and restoration of this function is required according to the following additional condition:

$$\int_0^l u(x,t)dx = r(t). \tag{6}$$

where $r(t)$ is given function.

Thus, the problem is to determine the functions $u(x,t)$ and $q(t)$ that satisfy equation (1) and conditions (2)-(6). The problem posed belongs to the class of inverse problems associated with the restoration of the right-hand sides of partial differential equations [2], [9].

3. Method for Solving the Problem

First, we reduce problem (1)-(6) to a problem with local conditions.

Let us integrate equation (1) on the segment $[0, x]$ with respect to x. Performing integration by parts and taking into account condition (4), we obtain

$$\frac{\partial^2}{\partial t^2}\int_0^x u(\xi,t)d\xi = \nu(t)\frac{\partial u(x,t)}{\partial x} - \nu(t)\theta(t) + \lambda(t)\int_0^x u(\xi,t)d\xi + q(t)\int_0^x F(\xi,t)d\xi.$$

By designating

$$\int_0^x u(\xi,t)d\xi = w(x,t),$$

we write the last integral relation in the form

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \nu(t)\frac{\partial^2 w(x,t)}{\partial x^2} - \nu(t)\theta(t) + \lambda(t)w(x,t) + q(t)f(x,t), 0 < x < l, \quad 0 < t \le T, \quad (7)$$

where $f(x,t) = \int_0^x F(\xi,t)d\xi$.

For equation (7), we will have the following initial conditions

$$w(x,0) = \varphi(x), \quad (8)$$

$$\frac{\partial w(x,0)}{\partial t} = \psi(x), \quad (9)$$

and the boundary conditions

$$w(0,t) = 0, \quad (10)$$

$$\frac{\partial w(l,t)}{\partial x} = p(t), \quad (11)$$

$$w(l,t) = r(t), \quad (12)$$

where $\varphi(x) = \int_0^x \varphi_1(\xi)d\xi$, $\psi(x) = \int_0^x \psi_1(\xi)d\xi$.

Let us construct a difference analogue of the differential problem (7) - (12). For this purpose, we introduce a uniform difference grid

$$\overline{\omega} = \{(x_i, t_j) : x_i = i\Delta x, \quad t_j = j\Delta t, \quad i = 0,1,2,...n, \quad j = 0,1,2,...,m\}$$

in a rectangular area $\{0 \le x \le l, \quad 0 \le t \le T\}$ with steps $\Delta x = l/n$ by variable x $\Delta t = T/m$ by time t. To the nonlinear equation (7) at the internal nodes of the grid $\overline{\omega}$, we associate the implicit difference scheme

$$\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} = \nu^{j+1}\frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{\Delta x^2} - \nu^{j+1}\theta^{j+1} + \lambda^{j+1}w_i^{j+1} + q^{j+1}f_i^{j+1},$$

$$i = \overline{1, n-1}, j = \overline{0, m-1}.$$

We write difference analogs of the initial and boundary conditions (8)-(12) in the form

$$w_i^0 = \varphi_i, \frac{w_i^1 - w_i^0}{\Delta t} = \psi_i,$$

$$w_0^{j+1} = 0, \frac{w_n^{j+1} - w_{n-1}^{j+1}}{\Delta x} = p^{j+1},$$

$$w_n^{j+1} = r^{j+1},$$

where

$$w_i^j \approx w(x_i, t_j), \quad \varphi_i = \varphi(x_i), \quad \nu^j = \nu(t_j), \quad \psi_i = \psi(x_i), \quad r^j = r(t_j),$$

$$\theta^j = \theta(t_j), \quad p^j = p(t_j), \quad f_i^j = f(x_i, t_j), \quad \lambda^j = \lambda(t_j), \quad q^j \approx q(t_j).$$

It should be noted that for the direct problem, this difference scheme has the first order of accuracy both in space and in time with an error $O(\Delta x, \Delta t)$ and is unconditionally stable.

We transform the resulting system of difference equations to the form

$$a_i w_{i-1}^{j+1} - c_i w_i^{j+1} + b_i w_{i+1}^{j+1} = -d_i^j - q^{j+1} f_i^{j+1}, \quad i = \overline{1, n-1}, \quad j = \overline{0, m-1}, \tag{13}$$

$$w_i^0 = \varphi_i, \quad w_i^1 = w_i^0 + \psi_i \Delta t, \tag{14}$$

$$w_0^{j+1} = 0, \quad w_n^{j+1} = w_{n-1}^{j+1} + p^{j+1} \Delta x, \tag{15}$$

$$w_n^{j+1} = r^{j+1}, \tag{16}$$

where

$$a_i = \frac{\nu^{j+1}}{\Delta x^2}, \quad b_i = \frac{\nu^{j+1}}{\Delta x^2}, \quad c_i = a_i + b_i + \frac{1}{\Delta t^2} - \lambda^{j+1}, \quad d_i^j = \frac{2w_i^j - w_i^{j-1}}{\Delta t^2} - \nu^{j+1} \theta^{j+1}.$$

Difference problem (13) - (16) is a system of linear algebraic equations in which the unknowns are approximate values of the sought functions $w(x,t)$ and $q(t)$ at the internal nodes of the difference grid, i.e. w_i^{j+1}, q^{j+1} $i = \overline{1, n}$, $j = \overline{0, m-1}$. To solve this system, we use the approach proposed in [5]. We represent the solution of system (13) - (15) for each fixed value j in the form

$$w_{i+1}^{j+1} = \alpha_{i+1} w_i^{j+1} + \beta_{i+1}, \quad i = 0, 1, 2, \ldots, n-1, \tag{17}$$

where $\alpha_{i+1}, \beta_{i+1}$ so far unknown coefficients. Let's write a similar expression for w_i^{j+1}

$$w_i^{j+1} = \alpha_i w_{i-1}^{j+1} + \beta_i.$$

Substituting expressions w_i^{j+1}, w_{i-1}^{j+1} in equation (13), we obtain the following formulas for determining the coefficients α_i, β_i:

$$\alpha_i = \frac{a_i}{c_i - \alpha_{i+1} b_i}, \tag{18}$$

$$\beta_i = \frac{b_i \beta_{i+1} + d_i^j + q^{j+1} f_i^{j+1}}{c_i - \alpha_{i+1} b_i}, \quad i = n-1, n-2, ..., 1.$$

The initial values of these coefficients are found from the requirement of equivalence of representation (17) to the second equation (15) for $i = n - 1$

$$\alpha_n = 1, \quad \beta_n = p^{j+1} \Delta x.$$

Obviously, having determined α_n, the remaining values of the coefficients α_i, $i = n - 1, n - 2, ..., 1$, can be successively found by formula (18).

The nonlinear equation for β_i transforms to the form

$$\beta_i = \frac{b_i}{c_i - \alpha_{i+1} b_i} \beta_{i+1} + \frac{d_i^j}{c_i - \alpha_{i+1} b_i} + \frac{f_i^{j+1}}{c_i - \alpha_{i+1} b_i} q^{j+1}$$

or

$$\beta_i = s_i \beta_{i+1} + y_i + z_i q^{j+1},$$

where $s_i = \frac{b_i}{c_i - \alpha_{i+1} b_i}$, $y_i = \frac{d_i^j}{c_i - \alpha_{i+1} b_i}$, $z_i = \frac{f_i^{j+1}}{c_i - \alpha_{i+1} b_i}$.

Using elementary calculations, we transform the last relation for β_i to the form

$$\beta_i = \tilde{\beta}_i + \tilde{z}_i q^{j+1}, \quad i = 1, 2, ..., n - 1, \tag{19}$$

where the variables $\tilde{\beta}_i$ and $\tilde{z}_i$ are solutions of the following two independent first-order difference problems:

$$\tilde{\beta}_i = s_i \tilde{\beta}_{i+1} + y_i, \quad \tilde{\beta}_n = \beta_n, \tag{20}$$

$$\tilde{z}_i = s_i \tilde{z}_{i+1} + z_i, \quad \tilde{z}_n = 0, \quad i = n - 1, n - 2, ..., 1. \tag{21}$$

Now we will find the relationship between q^{j+1} and w_n^{j+1} in an explicit form. For this, we write representation (17) $i = n - 1$

$$w_n^{j+1} = \alpha_n w_{n-1}^{j+1} + \beta_n.$$

Substituting here the expression for w_{n-1}^{j+1}, i.. $w_{n-1}^{j+1} = \alpha_{n-1} w_{n-2}^{j+1} + \beta_{n-1}$, we have

$$w_n^{j+1} = \alpha_n \alpha_{n-1} w_{n-2}^{j+1} + \alpha_n \beta_{n-1} + \beta_n.$$

Further, substituting into the last equation the expressions for w_{n-2}^{j+1}, w_{n-3}^{j+1}, ..., w_1^{j+1}, we obtain a formula in which w_n^{j+1} expressed through w_0^{j+1}

$$w_n^{j+1} = w_0^{j+1} \prod_{i=1}^{n} \alpha_i + \sum_{i=1}^{n-1} \beta_i \prod_{k=i+1}^{n} \alpha_k + \beta_n. \tag{22}$$

Now, substituting relation (19) into equation (22), we obtain the required relationship between w_n^{j+1} and q^{j+1}

$$w_n^{j+1} = w_0^{j+1} \prod_{i=1}^{n} \alpha_i + \sum_{i=1}^{n-1} \tilde{\beta}_i \prod_{k=i+1}^{n} \alpha_k + q^{j+1} \sum_{i=1}^{n-1} \tilde{z}_i \prod_{k=i+1}^{n} \alpha_k + \beta_n.$$

From the resulting equation, you can find the approximate value of the desired function $q(t)$ for $t = t_{j+1}$

$$q^{j+1} = \frac{w_n^{j+1} - w_0^{j+1} \prod_{i=1}^{n} \alpha_i - \sum_{i=1}^{n-1} \tilde{\beta}_i \prod_{k=i+1}^{n} \alpha_k - \beta_n}{\sum_{i=1}^{n-1} \tilde{z}_i \prod_{k=i+1}^{n} \alpha_k}.$$

Taking into account conditions (15), (16), the last formula can be written in the form

$$q^{j+1} = \frac{r^{j+1} - \sum_{i=1}^{n-1} \tilde{\beta}_i \prod_{k=i+1}^{n} \alpha_k - p^{j+1} \Delta x}{\sum_{i=1}^{n-1} \tilde{z}_i \prod_{k=i+1}^{n} \alpha_k}. \tag{23}$$

Defining q^{j+1} by formula (23), using the recurrent formula (17), one can sequentially determine w_1^{j+1}, $w_2^{j+1}, ..., w_{n-1}^{j+1}$. When moving to the next time layer, the described calculation procedure is repeated again.

Thus, the computational algorithm for solving the inverse problem (1)-(6) to restore the value of the function $q(t)$ for each discrete value of the time variable t_j, $j = 1, 2, ..., m$ is based on:

solving two linear difference problems of the first order (20), (21) with respect to auxiliary variables $\tilde{\beta}_i$ and $\tilde{z}_i$, $i = \overline{1,n}$;

determining the values of variables α_i, β_i, $i = \overline{1,n}$, by formulas (18), (19);

definition q^{j+1} from (23);

using representation (17) to calculate w_i^{j+1}, $i = \overline{1,n}$.

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