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A MIXED PROBLEM FOR A ONE-DIMENSIONAL VISCOELASTICITY EQUATION WITH NON-STATIONARY CONJUGATION CONDITIONS

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Abstract. *The problem of a longitudinal impact on a piecewise homogeneous semi-infinite rod consisting of viscoelastic parts is studied. Introducing non-stationary dynamic regularization under conjugation conditions, we prove the well-posedness of the problem under consideration*

Keywords: nonstationary regularization, composite linear visco-elastic bar, mixed problem,weak solution, strong solution

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1. Introduction

The study of the dynamics of wave propagation arising from a longitudinal impact on a piecewise-homogeneous semi-infinite rod, is an urgent problem, which is of great practical interest [2]-[4], [6], [8], [9], [12]-[14].The initial-boundary value problem of longitudinal impact on a piecewise-homogeneous semi-infinite rod, consisting of a semi-infinite elastic part and a viscoelastic part of finite length, the hereditary properties of which are described by linear integral relations with an arbitrary difference kernel was investigated in the work [1].

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In this paper, we study the problem of a longitudinal impact on a piecewise homogeneous semi-infinite rod consisting of viscoelastic parts, the hereditary properties of which are described by linear integral equations with arbitrary difference kernels when an impulse load is applied to the end of the rod. First of all, the considered problem is regulated by the problem with a dynamic boundary condition and a dynamic conjugation condition. Further after that, the regularised problem is reduced to an integro-differential equation with operator coefficients in some extended functional space. Having studied the resulting equation, we obtain the necessary a priori estimates that allow to pass to the limit with respect to a small parameter.

2. Problem Statement and Basic Results

The problem of a longitudinal impact on a piecewise homogeneous semi-infinite rod consisting of viscoelastic parts is studied, assuming that the rod has a past history. The mathematical model of the stated problem is expressed in the following way:

$$\frac{1}{a_i^2}\ddot{u}_i(t,x)=\sigma_{ix}(t,x),\ t>0,\ x\in(h_{i-1},h_i)\,,\ t>0,\ i=1,...,m, \tag{1}$$

$$u_i(t,h_{i+1})=u_{i+1}(t,h_{i+1}),\ t>0,\ i=1,...,m-1, \tag{2}$$

$$\sigma_1(t,h_0)=f(t),\quad t>0, \tag{3}$$

$$\sigma_i(t,h_{i+1})-\sigma_{i+1}(t,h_{i+1})=0,\quad t>0,\ i=1,...m-1, \tag{4}$$

$$\sigma_m(t,h_m)=0,\quad t>0, \tag{5}$$

$$u_i\ (t,x)=\gamma_i(t,x)\ \ t\le 0,\quad x\in(h_{i-1},h_i)\,,\quad i=1,\dots,m, \tag{6}$$

where $h_0<h_1<h_2<...<h_m$, $a_i>0$, $i=1,...,m$, $u_i(t,x)$ deviation of the $i-$th component of the rod from the absisaxisat the point $x\in(h_{i-1},h_i)$ and at time t, $\dot{u}_i=u_{it}$, $\ddot{u}_i=u_{itt}$, $i=1,...,m$, $\Gamma(\cdot)$ is a real function and

$$\Gamma'(s)\le c\Gamma(s),\quad s\in\mathbb{R},\quad c>0. \tag{7}$$

Let $\sigma_i(t,x)$ be the stresses of the $i-$th component of the bar at point x, at time t. It is assumed that [2], [5], [11]

$$\sigma_i(t,x)=u_{ix}(t,x)-\int_{-\infty}^{t}\Gamma(t-\tau)u_{ix}(\tau,x)d\tau,\quad i=1,\dots,m. \tag{8}$$

Denote by $W_2^1(h_{i-1},h_i)$ and $W_2^2(h_{i-1},h_i)$ the following functional spaces:

$$W_2^1(h_{i-1},h_i)=\{v:\ v\in L_2(h_{i-1},h_i),\ v_x\ \in L_2(h_{i-1},h_i)\}\,,$$

$$W_2^2(h_{i-1},h_i)=\{v:\ v\in L_2(h_{i-1},h_i),\ v_{xx}\in L_2(h_{i-1},h_i)\}\,,$$

where $L_2(h_{i-1},h_i)$ be the space of measurable functions summable with a square in (h_{i-1},h_i), $i=1,\dots,m$.

Let's denote the set of continuous functions with values in some Banach space Y by $C([0,T];Y)$ and the set of weakly continuous functions with values in Y by $C_w([0,T];Y)$.

We denote the set of continuously differentiable functions with values in Y by $C^1([0,T];Y)$.

By a weak solution to problem (1)-(6) we mean such a functions $(u_1(t,x),...,u_m(t,x))$, that

– $u_i(\cdot)\in C_w\left([0,T];\ W_2^1(h_{i-1},h_i)\right),\ u_{it}(\cdot)\in C_w\left([0,T];\ L_2(h_{i-1},h_i)\right),\ i=1,2,\dots,m;$
– for all $\eta_i(\cdot)\in C_w\left([0,T];W_2^1(h_{i-1},h_i)\right),\ \eta_{it}(\cdot)\in C_w\left([0,T];\ L_2(h_{i-1},h_i)\right),$
$i=1,2,\dots,m,\ \eta_i(t,h_i)=\eta_{i+1}(t,h_i),\ t>0,\ i=1,2,\dots,m-1,\ \eta_m(t,h_m)=0$ the following equalities hold:

$$\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}u_{it}(T,x)\eta_i(T,x)dx+\sum_{i=1}^{m}\int_0^T\int_{h_{i-1}}^{h_i}\left[-u_{it}(t,x)\eta_{it}(x,t)+\sigma_i(t,x)\eta_{ix}(t,x)\right]dxdt=$$

$$=\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}\gamma_i(0,x)\eta_i(0,x)dx+\sum_{i=1}^{m}\int_0^T f(t)\eta_1(h_0,x)dx;$$

$$\lim_{t\to 0}\left\langle u_i(t,\cdot)-\gamma_i(0),\eta_i(t,\cdot)\right\rangle_{W_2^1(h_{i-1},h_i)}=0,\quad i=1,...,m.$$

Let $X\subset(-\infty,\ +\infty)$. We denote the following space by $\mathcal{C}_i^1(X)$ and $\mathcal{C}_i^2(X)$, respectively:

$$\mathcal{C}_i^1(X)=\{u: u\in C(X;W_2^1(h_{i-1},h_i)),\ u_t\in C(X;L_2(h_{i-1},h_i))\},$$

$$\mathcal{C}_i^2(X)=\{u: u\in C(X;W_2^2(h_{i-1},h_i)),u_t\in C(X;W_2^1(h_{i-1},h_i)),u_{tt}\in C(X;L_2(h_{i-1},h_i))\}.$$

The main goal of this paper is to prove the following theorem

Theorem 1. *Let the conditions (7) and (8) be satisfied, assume that* $\gamma_i(\cdot)\in\mathcal{C}_i^1(-\infty,0],\ i=1,...,m$ *and* $\gamma_i(t,h_i)=\gamma_{i+1}(t,h_i),\ i=1,...,m-1$. *Then the problem (1)-(6) has a unique weak solution* $(u_1(\cdot),...,u_m(\cdot))$, *where* $u_i(\cdot)\in\mathcal{C}_i^1[0,+\infty),\ i=1,...,m$.

Theorem 1 is proved using dynamic regularization in the boundary conditions. More precisely, in order to solve this problem, we will first investigate a mixed problem with a dynamic boundary condition and dynamic transmission conditions(see[1]).Thus, firstly we investigate the following problem:

$$\frac{1}{a_i^2}u_{itt}(t,x)=\sigma_{ix}(t,x),\ t>0,\ x\in(h_i,h_{i+1}),\ i=1,...,m, \tag{9}$$

$$u_i(t,h_i)=u_{i+1}(t,h_i),\ t>0,\ i=1,...,m-1, \tag{10}$$

$$\varepsilon u_{1tt}(t,h_0)-\sigma_1(t,h_0)+f(t)=0,\ t>0, \tag{11}$$

$$\varepsilon u_{itt}(t,h_i)+\sigma_i(t,h_i)-\sigma_{i+1}(t,h_i)=0,\ t>0,\ i=1,...m-1, \tag{12}$$

$$\sigma_m(t,h_m)=0,\ t>0, \tag{13}$$

$$u_i(t,x)=\gamma_i(t,x),\ t\le 0,\ x\in(h_i,h_{i+1}),\ i=1,2,\dots,m, \tag{14}$$

$$u_{it}(t,x)=\gamma_{it}(t,x),\ t\le 0,\ x\in(h_i,h_{i+1}),\ i=1,2,\dots,m, \tag{15}$$

where $\varepsilon > 0$ is a small parameter. The problem (9)-(15) are reduced to operator-differential equationin some Hilbert space.

The strong solution to problem (9)-(15) is a functions $(u_1(t,x),...,u_m(t,x))$, defined in the domain $(0,T)\times\bigcup_{i=1}^{m}(h_{i-1},h_i)$ such that $u_i(\cdot)\in L_\infty(0,T;W_2^2(h_{i-1},h_i))$, $u_{it}(\cdot)\in L_\infty\left(0,T;W_2^1(h_{i-1},h_i)\right), u_{itt}(\cdot)\in L_\infty(0,T;L_2(h_{i-1},h_i))$, $i=1,...,m$ and for almost all $(t,x)\in(0,T)\times(h_{i-1},h_i)$ the functions $u_i(\cdot)$, $i=1,...,m$ satisfy the equation (9), boundary and conjugation conditions (10)-(13) and initial conditions (14), (15).

By a weak solution to problem (9)-(15) we mean such functions $(u_1(t,x),...,u_m(t,x))$, that

– $u_i(\cdot)\in C_w\left([0,T];W_2^1(h_{i-1},h_i)\right),\ u_{it}(\cdot)\in C_w\left([0,T];L_2(h_{i-1},h_i))\right),\ i=1,2,\ldots,m;$

– for all $\eta_i(\cdot)\in C_w\left([0,T];\ W_2^1(h_{i-1},h_i)\right), \eta_{it}(\cdot)\in C_w^1\left([0,T];\ L_2(h_{i-1},h_i)\right)$,
$i=1,2,\ldots,m,\ \eta_i(t,h_i)=\eta_{i+1}(t,h_i),\ i=1,2,\ldots,m-1,\ \eta_m(t,h_m)=0$ the following equalities hold:

$$\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}u_{it}(T,x)\eta_i(T,x)dx+\sum_{i=1}^{m}\int_0^T\int_{h_{i-1}}^{h_i}\left[-u_{it}(t,x)\eta_{it}(t,x)+\sigma_i(t,x)\eta_{ix}(t,x)\right]dxdt=$$

$$=\varepsilon u_{1t}(T,h_0)\eta_1(T,h_0)+\varepsilon\sum_{i=1}^{m}u_{it}(T,h_i)\eta_i(T,h_i)\ -$$

$$-\varepsilon\int_0^T u_{1t}(t,h_0)\eta_{1t}(t,h_0)dt-\varepsilon\sum_{i=1}^{m}\int_0^T -u_{it}(t,h_i)\eta_{it}(t,h_i)dt=$$

$$=\ \sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}\gamma_i(0,x)\eta_i(0,x)dx+\sum_{i=1}^{m}\int_0^T f(t)\eta_1(h_0,x)dx\ +$$

$$+\varepsilon\gamma_{1t}(0,h_0)\eta_1(0,h_0)+\varepsilon\sum_{i=1}^{m}\gamma_{it}(0,h_i)\eta_i(0,h_i)\ ;$$

$$\lim_{t\to 0}\langle u_i(t,.)-\gamma_i(0,\cdot),\eta_i(t,\cdot)\rangle_{W_2^1(h_{i-1},h_i)}=0,\ i=1,...,m.$$

For the problem (9)-(15), the following solvability theorem is true.

Theorem 2. *Suppose that conditions (7), (8) are fulfilled, $\gamma_i(\cdot)\in\mathcal{C}_i^1(-\infty,0]$, $i=1,...,m$, and $\gamma_i\ (t,h_i)=\gamma_{i+1}\ (t,h_i)$, $i=1,...,m-1$. Then for any $\varepsilon>0$ the problem (9)-(15) has a unique solution $(u_1(.),...,\ u_m(\cdot))$, where $u_i(\cdot)\in\mathcal{C}_i^1[0,+\infty)$, $i=1,...,m$.*

If additionally $\gamma_i(\cdot)\in\mathcal{C}_i^2(-\infty,0]$, $i=1,...,m$, then $u_i(\cdot)\in\mathcal{C}_i^2[0,+\infty)$, $i=1,...,m$. Moreover, weak solutions are the limit of strong solutions. In other words, if $(u_1(\cdot),...,u_m(\cdot))$ is a weak solution of the problem (9)-(15) with initial data's $\gamma_i(\cdot)\in\mathcal{C}_i^1(-\infty,0]$, $i=1,...,m$, where $\gamma_i\ (t,h_i)=\gamma_{i+1}\ (t,h_i)$, $i=1,...,m-1$, then there exist initial data's $\gamma_{i,n}(\cdot)\in\mathcal{C}_i^1(-\infty,0]$, $i=1,...,m$, where $\gamma_{i,n}(t,h_i)=\gamma_{i+1,n}(t,h_i)$, $i=1,...,m-1$, such that $\gamma_{i,n}(\cdot)\to\gamma_i(\cdot)$ in $\mathcal{C}_i^1(-\infty,0]$, $i=1,...,m$, and the corresponding strong solution $(u_{1,n}(\cdot),...,u_{m,n}(\cdot))$ of the problem (9)-(15) with initial data's $u_{i,n}(0,x)=\gamma_{i,n}(0,x)$, $i=1,\ldots,m$, converges to the $(u_1(\cdot),...,u_m(\cdot))$ in the space $\prod_{i=1}^{m}\mathcal{C}_i^1[0\ ,+\infty)$.

3. Introduction of Some Notation and Preliminary Lemmas

Denote by H_0 and H_1 the following functional spaces:

$$H_0 = \{v : v = (v_1, v_2, ..., v_m)\,,\ v_i\ \in W_2^2(h_{i-1}, h_i),\ i = 1, ..., m, v_i(h_i) = v_{i+1}(h_i),$$
$$i = 1, ..., m-1\}\,,$$
$$H_1 = \{v = (v_1, v_2, ..., v_m)\,, v_i \in W_2^1\,(h_{i-1}, h_i)\,,\ \ i = 1, \dots, m, v_i\,(h_i) = v_{i+1}\,(h_i)\,,$$
$$i = 1, \dots, m-1, v_m\,(h_m) = 0\}\,.$$

Let $\mathbb{C}$ is a complex plane. We denote by

$$\mathcal{H} = \mathbb{C} \oplus \prod_{i=1}^{m} [L_2(h_{i-1}, h_i) \oplus\ \mathbb{C}]$$

the Hilbert space with a scalar product

$$\left\langle w^1, w^2 \right\rangle_{\mathcal{H}} = a_i^2 \sum_{i=1}^{m} \int_{h_{i-1}}^{h_i} v_i^1(x) v_i^2(x) dx + \frac{1}{\varepsilon} \sum_{i=0}^{m-1} \alpha_i^1 \alpha_i^2,$$

where $w^k = \left(\alpha_0^k, v_1^k\,(x)\,, \alpha_1^k, \dots, v_{m-1}^k\,(x)\,, \alpha_{m-1}^k, v_m^k\,(x)\right),\ v_i^k\,(\cdot) \in L_2\,(h_{i-1}, h_i)\,,\ i = 1, \dots, m, \alpha_i^k \in \mathbb{C}, i = 0, 1, ..., m,\ k = 1, 2.$

We denote by $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ the norm in the space $\mathcal{H}$. We also define spaces $\mathcal{H}_0$ as follows:

$$\mathcal{H}_0 = \left\{ w : w = (\varepsilon v_1(h_0), \frac{1}{a_1^2} v_1, \varepsilon v_1(h_1), \frac{1}{a_2^2} v_2,\ \varepsilon v_2(h_2), ..., \right.$$
$$\left. \frac{1}{a_{m-1}^2} v_{m-1}, \varepsilon v_m(h_{m-1}),\ \frac{1}{a_m^2} v_m),\ \text{where } w = (v_1, v_2, ..., v_m) \in H_0 \right\}.$$

Let $\mathcal{H}_1$= $[\mathcal{H}_0, \mathcal{H}]_{1/2}$ be an interpolation space between $\mathcal{H}_0$ and $\mathcal{H}$ of order $\frac{1}{2}$ (see[10]).

Lemma 1. *$\mathcal{H}_0$ is dense in $\mathcal{H}$.*

In the space $\mathcal{H}$ we define the linear operator A:

$$\begin{cases} D(A) = \mathcal{H}_0, \\ Aw = (-v_1(h_0), -v_{1_{xx}}, v_{1_x}(h_1) - v_{2_x}(h_1), -v_{2_{xx}}, v_{2_x}(h_2) - v_{3_x}(h_2),\ ..., \\ -v_{m-1_{xx}}, v_{m-1_x}(h_{m-1}) - v_{m_x}(h_m), -v_{m_{xx}}). \end{cases}$$

Then, in the space $\mathcal{H}$ problem (9)-(15) can be written in the following form:

$$\begin{cases} \ddot{w} + Aw + \int_{-\infty}^{t} \Gamma\,(t-\tau)\, Aw\,(\tau)\, d\tau = F\,(\tau)\,, \\ w\,(t) = w_0(t),\ w_t\,(0) = w_1(t),\ \ t < 0, \end{cases} \tag{16}$$

where $w_0\,(t) = (\gamma_1\,(t,x)\,, \dots, \gamma_m\,(t,x))\,,\ w_1(t) = (\gamma_{1t}\,(t,x)\,, \dots, \gamma_{mt}\,(t,x))\,, t < 0,\ x \in (h_{i-1}, h_i),\ i = 1, ..., m, F(t) = (0, f(t), 0, ..., 0).$

Lemma 2. *A is symmetric operator in $\mathcal{H}$.*

Lemma 3. *$R(A+\lambda I)=\mathcal{H}$ for some $\lambda\in\mathbb{R}$ and $\langle Aw,w\rangle_{\mathcal{H}}\geq 0$ for all $w\in D(A)$.*

By virtue of Lemmas 2 and 3, we obtain the following statement.

Proposition. *A is a self-adjoint positive operator in $\mathcal{H}$.*

Following Dafermos [5],we will add a new variable $\eta$to the system which corresponds to the relative displacement history. Let us define

$$\eta=\eta^t(s)=w(t)-w(t-s),\ \ s\in\mathbb{R},$$

$$\eta^0(s)=w_0(0)-w_0(-s),\ \ s\in\mathbb{R}.$$

By differentiation we have

$$\eta_t=w_t-\eta_s. \tag{17}$$

Using variable substitution, we find that

$$\int_{-\infty}^{t}\mathit{\Gamma}(t-s)Aw(s)ds=\int_0^{+\infty}\mathit{\Gamma}(s)ds\cdot Aw(t)-\int_0^{+\infty}\mathit{\Gamma}(s)A\eta(s)ds. \tag{18}$$

It follows from (16),(17) and (18) that (w,η) is a solution to the following Cauchy problem:

$$\begin{cases} w_{tt}+(1+\ \int_0^{+\infty}\mathit{\Gamma}(s)ds)\cdot Aw(t)+\int_0^{+\infty}\mathit{\Gamma}(s)A\eta(s)ds=F(t),\\ \eta_t+\eta_s=w_t,\\ w(0)=w_0,\ \ w_t(0)=w_1,\\ \eta^0(s)=w_0(0)-w_0(-s),\ \ s\in\mathbb{R},\end{cases} \tag{19}$$

where $w_0=w_0\left(0\right),w_1=w_{0t}\left(0\right).$ We make the substitution $v_1=w,v_2=w_t,v_3=\eta$ and introduce the Hilbert space $W=\mathcal{H}_1\times\mathcal{H}\times L_{2,\mathit{\Gamma}(s)}(0,\infty;\mathcal{H}_1)$ with a scalar product

$$\left\langle z^1,z^2\right\rangle_W=[1+\ \int_0^{+\infty}\mathit{\Gamma}(s)ds]\ \left\langle A^{1/2}v_1^1,A^{1/2}v_1^2\right\rangle_{\mathcal{H}}+\left\langle v_2^1,v_2^2\right\rangle_{\mathcal{H}}+$$

$$+\int_0^{+\infty}\mathit{\Gamma}(s)\Big\langle A^{1/2}\eta^1(s),A^{1/2}\eta^2(s)\Big\rangle_{\mathcal{H}}ds,$$

where $z^k=(v_1^k,\ v_2^k,\ v_3^k),\ k=1,2.$ We also define linear operator $\mathcal{A}$:

$$\mathcal{A}z=(v_2,-[1+\ \int_0^{+\infty}\mathit{\Gamma}(s)ds)]Av_1+\int_0^{+\infty}\mathit{\Gamma}(s)Av_3(t,s)ds,\ -v_{3s}+\ v_2\ ),$$

where

$$z\ \in D(\mathcal{A})=\{z:\ z=\ (v_1,v_2,v_3)\ ,\ v_1\in\mathcal{H}_0,\ v_2\in\mathcal{H}_1,$$

$$v_3\in\ L_{2,\mathit{\Gamma}(s)}(0,\infty;\mathcal{H}_0),v'_3\in L_{2,\mathit{\Gamma}(s)}(0,\infty;\mathcal{H}_1),\ v_3(0)=0\big\}\,.$$

In the W space, problem (19) is reduced to the Cauchy problem

$$\begin{cases} z_t(t) = \mathcal{A}z(t) + \Phi(t), \\ z(0) = z_0, \end{cases} \tag{20}$$

where $\Phi(t) = (0, F(t), 0)$, $z_0 = (w_0, w_1, \eta^0(s))$.

Lemma 4. *Suppose that condition (7) is satisfied, then $\mathcal{A}$ is a dissipative operator in W.*

Lemma 5. *There is a point $\lambda_0 > 0$ such that $R(\mathcal{A} - \lambda_0 I) = \mathcal{H}$.*

Lemmas 4 and 5 imply the following statement (see[7]).

Theorem 3. *Suppose that condition (7) is fulfilled. Then for any $z_0 \in \mathcal{H}$, the problem (20) has a unique solution $z(\cdot) \in C([0,\infty), W)$. If $z_0 \in D(\mathcal{A})$, then $z(\cdot) \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), W)$.*

If $z_0 \in W$, then the corresponding weak solutions $z(\cdot) \in C([0,\infty), W)$ of the problem (20) are the limits of the strong solutions $z_n(\cdot) \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), W)$ in the space $C([0,\infty), W)$, where $z_n(0) = z_{0n} \in D(\mathcal{A})$ and $z_{0n} \to z_0$ in W [7].

This implies the assertion of Theorem 1.

4. Proof of Theorem 2

Obviously, $u_{i,n}(t,x), i = 1, \ldots, m$, depends on $\varepsilon = \frac{1}{n}$. We will prove that $u_{i,n}(t,x)$ has a limit at $n \to +\infty$ and the limit function is a solution to the problem (1)-(6).

At first, we assume $u_i(\cdot) \in \mathcal{C}_i^1[0,+\infty)$, $i = 1, ..., m$. In view of Theorem 1, all subsequent operations are justified.

Let T be a positive number, $\gamma_{i,n}(\cdot) \in \mathcal{C}_i^1(-\infty, 0]$, $i = 1, ..., m$, where $\gamma_{i,n}(t,h_i) = \gamma_{i+1,n}(t,h_i)$, $i = 1, ..., m-1$, such that

$$\gamma_{i,n}(\cdot) \to \gamma_i(\cdot) \text{ in } \mathcal{C}_i^1(-\infty, 0], \; i = 1, ..., m, \tag{21}$$

and $(u_{1,n}(\cdot), ..., u_{m,n}(\cdot))$ is the corresponding strong solution of the problem

$$\frac{1}{a_i^2} u_{i,ntt}(t,x) = \sigma_{i,nx}(t,x), \; t>0, \; x \in (h_i, h_{i+1}), \; i = 1, ..., m-1, \tag{22}$$

$$u_{i,n}(t,h_i) = u_{i+1,n}(t,h_i), \; t>0, \; i = 1, ..., m-1, \tag{23}$$

$$\frac{1}{n} u_{1,ntt}(t,h_0) - \sigma_{1,n}(t,h_0) + f(t) = 0, \; t>0, \tag{24}$$

$$\frac{1}{n} u_{i,ntt}(t,h_i) + \sigma_{i,n}(t,h_i) - \sigma_{i+1,n}(t,h_i) = 0, \; t>0, \; i = 1, ...m-1, \tag{25}$$

$$\sigma_{m,n}(t,h_m) = 0, \quad t > 0, \tag{26}$$

$$u_{i,n}(t,x) = \gamma_{i,n}(t,x), \; t \leq 0, \; x \in (h_i, h_{i+1}), \; i = 1, 2, \ldots, m, \tag{27}$$

$$u_{i,nt}(t,x) =\gamma_{i,nt}(t,x), \quad t \leq 0, \quad x \in (h_i, h_{i+1}), \quad i = 1,2,\dots,m, \tag{28}$$

where

$$\sigma_{i,n}(t,x) = u_{i,nx}(t,x) - \int_{-\infty}^{t} \Gamma(t-\tau) u_{i,nx}(\tau,x)d\tau, \quad i = 1,\dots,m.$$

We multiply equation (22) by $u_{i,nt}(t,x)$ and integrate with respect to domain $[0,T] \times [h_{i-1},h_i]$, $i = 1,...,m$. Summing the obtained equations and integrating by parts, we obtain the following equality:

$$\frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{a_i^2}\int_{h_{i-1}}^{h_i} |u_{i,nt}(t,x)|^2 dx + \int_{h_{i-1}}^{h_i} |u_{i,nx}(t,x)|^2 dx\right] =$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{a_i^2}\int_{h_{i-1}}^{h_i} |\gamma_{i,nt}(0,x)|^2 dx + \int_{h_{i-1}}^{h_i} |\gamma_{i,nx}(0,x)|^2 dx\right] + \Phi(t) + \Psi(t),$$

where

$$\Phi(t) = -\sum_{i=1}^{m}\left[\int_0^t u_{i,nx}(s,h_i)u_{i,ns}(s,h_i)ds - \int_0^t u_{i,nx}(s,h_{i-1})u_{i,ns}(s,h_{i-1})ds\right],$$

$$\Psi(t) = \sum_{i=1}^{m}\int_0^t\int_0^s\int_{h_{i-1}}^{h_i} \Gamma(s-\tau)u_{i,nxx}(\tau,x)d\tau \cdot u_{i,ns}(s,x)dxds =$$

$$= \sum_{i=1}^{m}\int_0^t\int_0^s \Gamma(s-\tau)u_{i,nx}(\tau,h_i)u_{i,ns}(s,h_i)d\tau ds -$$

$$-\sum_{i=1}^{m}\int_0^t\int_0^s \Gamma(s-\tau)u_{i,nx}(\tau,h_{i-1})u_{i,ns}(s,h_{i-1})d\tau ds +$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i} \Gamma(t-\tau)u_{i,nx}(\tau,x)u_{i,nx}(t,x)dxd\tau +$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i} \Gamma(0)\, u_{i,nx}(s,x) \cdot u_{i,nx}(s,x)\, dxds +$$

$$+\sum_{i=1}^{m}\int_0^t\int_0^s\int_{h_{i-1}}^{h_i} \Gamma'(s-\tau)\, u_{i,nx}(\tau,x)\, u_{i,nx}(s,x)d\tau dxds.$$

By virtue of (23)-(26), we have

$$\frac{1}{n}u_{1,ntt}(s,h_0) - u_{1,nx}(s,h_i) +$$

$$+\int_0^s \Gamma(s-\tau)\, u_{1,nx}(\tau,h_i)\, d\tau \;+\; f(t) = 0, \;\; s > 0;$$

$$u_{i,nx}(s,h_i) + \int_0^s \Gamma(s-\tau)\, u_{i,nx}(\tau,h_i)\, d\tau = u_{i+1,nx}(s,h_i) +$$

$$+\int_0^s \Gamma\left(s-\tau\right)u_{i+1,nx}\left(\tau,h_i\right)d\tau,\quad s>0,\quad i=1,...,m-1.$$

It follows from (22)-(28) that

$$G(t)=\Phi(t)+\Psi(t)=-\frac{1}{n}\sum_{1=1}^{m}\int_0^t u_{i,nss}(s,h_i)u_{i,ns}(s,h_i)ds+$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma(t-\tau)u_{i,nx}(\tau,x)u_{i,nx}(t,x)dxd\tau+$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma\left(0\right)u_{i,nx}\left(s,x\right)\cdot u_{i,nx}\left(s,x\right)dxds+$$

$$+\sum_{i=1}^{m}\int_0^t\int_0^s\int_{h_{i-1}}^{h_i}\Gamma'\left(s-\tau\right)u_{i,nx}\left(\tau,x\right)u_{i,nx}(s,x)d\tau dxds.$$

Taking into account (11) and (25) in (21) we get that

$$\frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{a_i^2}\int_{h_{i-1}}^{h_i}|u_{i,nt}(t,x)|^2dx+\int_{h_{i-1}}^{h_i}|u_{i,nx}(t,x)|^2dx\right]+\frac{1}{2n}\sum_{i=1}^{m}|u_{i,ns}(t,h_i)|^2ds\ =$$

$$=\frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{a_i^2}\int_{h_{i-1}}^{h_i}|\gamma_{i,nt}(0,x)|^2dx\ +\int_{h_{i-1}}^{h_i}|\gamma_{i,nx}(0,x)|^2dx\right]+$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma(t-\tau)u_{i,nx}(\tau,x)u_{i,nx}(t,x)dxd\tau+$$

$$+\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma\left(0\right)u_{i,nx}\left(s,x\right)\cdot u_{i,nx}\left(s,x\right)dxds+$$

$$+\sum_{i=1}^{m}\int_0^t\int_0^s\int_{h_{i-1}}^{h_i}\Gamma'\left(s-\tau\right)u_{i,nx}\left(\tau,x\right)u_{i,nx}(s,x)d\tau dxds+\int_0^t u_{1,ns}(s,h_0)f(s)ds.\quad (29)$$

Applying Hölder's inequality, we obtain the following inequalities:

$$\left|\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma\left(t-\tau\right)u_{i,nx}\left(\tau,x\right)u_{i,nx}\left(t,x\right)dxd\tau\right|\le$$

$$\le c_\delta\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}|u_{ix}(\tau,x)|^2dxd\tau+\delta\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}|u_{ix}(t,x)|^2dx,\quad (30)$$

$$\left|\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\Gamma\left(0\right)u_{i,nx}\left(s,x\right)\cdot u_{i,nx}\left(s,x\right)dxds\right|\le$$

$$\leq c\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\left|u_{i,nx}\left(s,x\right)\right|^2dxds, \tag{31}$$

$$\left|\sum_{i=1}^{m}\int_0^t\int_0^s\int_{h_{i-1}}^{h_i}\Gamma'\left(s-\tau\right)u_{i,nx}\left(\tau,x\right)u_{i,nx}(s,x)d\tau dxds\right|\leq$$

$$\leq c\sum_{i=1}^{m}\int_0^t\int_{h_{i-1}}^{h_i}\left|u_{i,nx}\left(s,x\right)\right|^2dxds, \tag{32}$$

$$\left|\int_0^t u_{1,ns}(s,h_0)f(s)ds\right|\leq\int_0^t\left|u_{1,ns}(s,h_0)\right|^2ds+\int_0^t\left|f(s)\right|^2ds. \tag{33}$$

Applying the Gronwall lemma, from (29)-(33), we obtain a'priori estimation

$$\sum_{i=1}^{m}\left[\int_{h_{i-1}}^{h_i}\left|u_{i,nt}(t,x)\right|^2dx\ +\int_{h_{i-1}}^{h_i}\left|u_{i,nx}(t,x)\right|^2dx\right]+$$

$$+\frac{1}{n}\left|u_{1,nt}(t,h_0)\right|^2+\frac{1}{n}\sum_{i=0}^{m}\left|u_{i,nt}(t,h_i)\right|^2\leq c,$$

where $c>0$ does not depend on n.

By the Lions-Aubin lemma, there exists a subsequence of $(u_{1,n_k},...,u_{m,n_k})$ still denoted as $(u_{1,n},...,u_{m,n})$ such that

$$u_{i,n}(\cdot)\to u_i(\cdot)\text{ weak star in }L_\infty(0,T;W_2^1(h_{i-1},h_i)),\quad i=1,...,m, \tag{34}$$

$$u_{i,nt}(\cdot)\to u_{it}(\cdot)\text{ weak star in }L_\infty(0,T;L_2(h_{i-1},h_i)),\quad i=1,...,m, \tag{35}$$

$$\frac{1}{n}u_{1,nt}(t,h_0)\to 0\text{ weak star in }L_\infty(0,T), \tag{36}$$

$$\frac{1}{n}u_{i,nt}(t,h_i)\to 0\text{ weak star in }\ L_\infty(0,T),\quad i=1,...,m, \tag{37}$$

$$\frac{1}{n}u_{1,nt}(T,h_0)\to 0, \tag{38}$$

$$\frac{1}{n}u_{i,nt}(T,h_i)\to 0,\quad i=1,...,m. \tag{39}$$

Let us write the equation (9) for the $(u_{1,n},\ldots,u_{m,n})$, multiply both sides of (9) by $\eta_i\left(\cdot\right)\in C_w\left(\left[0,T\right];\ W_2^1\left(h_{i-1},h_i\right)\right)$, where $\eta_{it}\left(\cdot\right)\in C_w\left(\left[0,T\right];L_2\left(h_{i-1},h_i\right)\right)$, $i=1,2,\ldots,m$, $\eta_i(t,h_i)=\eta_{i+1}(t,h_i)$, $i=1,2,\ldots,m-1$, $\eta_m(t,h_m)=0$ and integrate over the region $(h_{i-1},,h_i)\times[0,T]$, $i=1,...,m$. Then, after simple transformations, we obtain the equality

$$\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}u_{i,nt}(T,x)\eta_i(T,x)dx+$$

$$+\sum_{i=1}^{m}\int_0^T\int_{h_{i-1}}^{h_i}\left[-u_{i,nt}(t,x)\eta_{it}(t,x)+\sigma_{i,n}(t,x)\eta_{ix}(t,x)\right]dxdt=$$

$$= \frac{1}{n} u_{1,nt}(T, h_0)\eta_1(T, h_0) + \frac{1}{n}\sum_{i=1}^{m} u_{i.nt}(T, h_i)\eta_i(T, h_i) - \frac{1}{n}\int_0^T u_{1.nt}(t, h_0)\eta_{1t}(t, h_0)dt -$$

$$-\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i} \gamma_{i,n}(0, x)\eta_i(0, x)dx + \sum_{i=1}^{m}\int_0^T f(t)\eta_1(h_0, x)dx -$$

$$- \frac{1}{n}\gamma_{1,nt}(0, h_0)\eta_{1,n}(0, h_0) + \frac{1}{n}\sum_{i=1}^{m}\gamma_{i,nt}(0, h_i)\eta_i(0, h_i) \ . \qquad (40)$$

Taking into account (34)-(39), we pass to the limit in (40) as $n \to \infty$. As a result, we see that $(u_1, ..., u_m)$ is a weak solution to the problem (9)-(15).

5. Proof of Lemmas

Proof of Lemma 1. Assume that

$$w = \left(\alpha_0, v_1, \alpha_1, v_2, \alpha_2, ..., v_{m-1}, \alpha_{m-1}, v_m\right) \in \mathcal{H}.$$

Consider the following functions:

$$\zeta_i^0(x) = \frac{h_i - x}{h_i - h_{i-1}}\alpha_{i-1} + \frac{x - h_{i-1}}{h_i - h_{i-1}}\alpha_i, \quad x \in [h_i, h_{i+1}], \quad i = 1, ..., m-1,$$

$$\zeta_i^0(h_i) = \zeta_{i+1}^0(h_i) = \alpha_i, \quad i = 1, ..., m-1,$$

$$\zeta_m^0(x) = \frac{1}{1 + (x - h_{m-1})^2}\alpha_{m-1}, x \in [h_{m-1}, h_m), \quad \zeta_m^0(h_m) = v_m.$$

Let $v = (v_1, v_2, ..., v_m) \in H_0$. Consider the function

$$z = (z_1, z_2, ..., z_m) = \left(v_1 - \zeta_1^0, v_2 - \zeta_2^0, ..., v_m - \zeta_m^0\right).$$

It is obvious that$z \in \prod_{i=0}^{m-1} L_2(h_i, h_{i+1})$. On the other hand

$$\overline{\prod_{i=0}^{m-1} D(h_i, h_{i+1})} = \prod_{i=0}^{m-1} L_2(h_i, h_{i+1}),$$

where $D(h_i, h_{i+1})$, $i = 1, 2, \ldots, m$, is space of infinitely differentiable finite functions. Therefore, for an arbitrary $z \in \prod_{i=0}^{m-1} L_2(h_i, h_{i+1})$, there exist the functions $\theta_i \in \prod_{i=1}^{m} D(h_i, h_{i+1})$, such that

$$\sum_{i=1}^{m} \|z_i - \theta_i\|_{L_2(h_i, h_{i+1})} < \varepsilon. \qquad (41)$$

By denoting $\widetilde{\theta}_i = \overset{0}{\zeta_i} + \theta_i$ from (41), we get

$$\sum_{i=1}^{m} \left\|v_i - \widetilde{\theta}_i\right\|_{L_2(h_i, h_{i+1})} < \varepsilon,$$

where $\widetilde{\theta}_i \in C^\infty[h_i, h_{i+1}]$, $\widetilde{\theta}_i(h_i) \ =\alpha_{i-1}, \quad i= 1, 2, ...,m.$

Thus

$$\| \left(\alpha_0, v_1, \alpha_1, v_2, \alpha_2, \ldots, v_{m-1}, \alpha_{m-1}, v_m, \right) -$$
$$-(\widetilde{\theta}_1(h_0), \widetilde{\theta}_1, \widetilde{\theta}_1(h_1), \widetilde{\theta}_2, \widetilde{\theta}_2(h_2), \ldots, \widetilde{\theta}_{m-1}, \widetilde{\theta}_{m-1}(h_{m-1}),\ \widetilde{\theta}_m)\|_{\mathcal{H}} < \varepsilon.$$

Proof of Lemma 2. Let $w, z \in D\left(A\right)$, where

$$w = \left(\varepsilon v_1(h_0), \frac{1}{a_1^2}v_1, \varepsilon v_1(h_1), \frac{1}{a_2^2}v_2,\ \varepsilon v_2(h_2), ..., \frac{1}{a_{m-1}^2}v_{m-1}, \varepsilon v_m(h_{m-1}),\ \frac{1}{a_m^2}v_m\right),$$

$$z = \left(\varepsilon z_1(h_0), \frac{1}{a_1^2}z_1, \varepsilon z_1(h_1), \frac{1}{a_2^2}z_2,\ \varepsilon z_2(h_2), \ldots, \frac{1}{a_{m-1}^2}z_{m-1}, \varepsilon z_m(h_m),\ \frac{1}{a_m^2}z_m\right),$$

$$(v_1, v_2, ..., v_m) \in H_0,\ v_i(h_i) = v_{i+1}(h_i),$$

$$(z_1, z_2, ..., z_m) \in H_0,\ z_i(h_i) = z_{i+1}(h_i),\ i = 1, 2, ..., m.$$

By virtue of the definition of the operator A and the scalar product in $\mathcal{H}$, we obtain that

$$\langle Aw, z\rangle_{\mathcal{H}} = -a_i^2 \sum_{i=1}^{m} \int_{h_{i-1}}^{h_i} v_{ixx}(x)\frac{1}{a_i^2}z_i(x)dx - \frac{1}{\varepsilon}v_{1x}(h_0)\cdot\varepsilon\cdot z_1(h_0) +$$

$$+\frac{1}{\varepsilon}\sum_{i=1}^{m-1}[v_{ix}(h_i) - v_{(i+1)x}(h_i)]\varepsilon z_i(h_i) = -\sum_{i=1}^{m}[v_{ix}(h_i)z_i(h_i) - v_{ix}(h_{i-1})z_i(h_{i-1})] +$$

$$+\sum_{i=1}^{m}\left[v_i\left(h_i\right)z_{ix}\left(h_i\right) - v_i\left(h_{i-1}\right)z_{ix}\left(h_{i-1}\right)\right] - \sum_{i=1}^{m}\int_{h_{i-1}}^{h_i} v_i(x)z_{ixx}(x)dx -$$

$$-v_{1x}(h_0)z_1(h_0) + \sum_{i=1}^{m-1}[v_{ix}(h_i) - v_{(i+1)x}(h_i)]z_i(h_i).$$

Taking into account the equality $v_m\left(h_m\right) = z_m\left(h_m\right) = 0$, we get that

$$\langle Aw, z\rangle_{\mathcal{H}} = -a_i^2\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}\frac{1}{a_i^2}v_i(x)z_{ixx}(x)dx +$$

$$-\frac{1}{\varepsilon}\varepsilon v_{1x}(h_0)z_1(h_0) + \frac{1}{\varepsilon}\sum_{i=1}^{m-1}[z_{ix}(h_i) - z_{(i+1)x}(h_i)]\varepsilon v_i(h_i) = \langle w, \mathcal{A}z\rangle_{\mathcal{H}}.$$

Proof of Lemma 3. Let

$$\vartheta = (\alpha_0,\ \vartheta_1(x), \alpha_1, ..., \vartheta_{m-1}\ (x), \alpha_{m-1}, \vartheta_m(x)) \in \mathcal{H}\ .$$

Let us consider the equation

$$Aw + \lambda w = \vartheta,\ \lambda > 0,$$

This equation is equivalent to the following boundary value problem

$$-v_{ixx}+\lambda\frac{1}{a_i^2}v_i=\vartheta_i(x),\ x\in[h_{i-1},h_i],\quad i=1,...,m, \tag{42}$$

$$-v_{1x}(h_0)+\lambda\varepsilon v_1(h_0)=\alpha_0, \tag{43}$$

$$v_{ix}(h_i)-v_{(i+1)x}(h_i)+\lambda\varepsilon v_i(h_i)=\alpha_i,\quad i=1,...,m-1, \tag{44}$$

$$v_i(h_i)=v_{i+1}(h_i),\ i=1,...,m-1. \tag{45}$$

Using the general theory of boundary value problems for linear differential equations, we can prove that problem (42)-(45) has a solution $(v_1,...,v_m)\in H_0$.

Let $w\in D(A)$, then by the definition of the operator A and the scalar product in $\mathcal{H}$ we obtain that

$$\langle Aw,w\rangle_{\mathcal{H}}=-a_i^2\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}v_{ixx}(x)\frac{1}{a_i^2}v_i(x)dx-\frac{1}{\varepsilon}v_{1x}(h_0)\varepsilon v_1(h_0)+$$

$$+\frac{1}{\varepsilon}\sum_{i=1}^{m-1}[v_{ix}(h_i)-v_{(i+1)x}(h_i)]\varepsilon v_i(h_i)=\sum_{i=1}^{m}[v_{ix}(h_i)v_i(h_i)-v_{ix}(h_{i-1})v_i(h_{i-1})]+$$

$$+\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}v_{ix}(x)v_{ix}(x)dx-v_{1x}(h_0)v_1(h_0)+$$

$$+\sum_{i=1}^{m-1}[v_{ix}(h_i)-v_{(i+1)x}(h_i)]v_i(h_i)=\sum_{i=1}^{m}\int_{h_{i-1}}^{h_i}|v_{ix}(x)|^2dx\geq 0.$$

Proof of Lemma 4. Let $z=(w,v,\eta))\in D(\mathcal{A})$, then

$$\langle\mathcal{A}z,z\rangle_W=\left(1+\int_0^{+\infty}\Gamma(s)ds\right)\left\langle A^{1/2}w,A^{1/2}v\right\rangle_{\mathcal{H}}+$$

$$+\left\langle-\left[1+\int_0^{+\infty}\Gamma(s)ds\right]Aw+\int_0^{+\infty}\Gamma(s)A\eta(x,s)ds\ ,v\right\rangle_{\mathcal{H}}+$$

$$+\int_0^{+\infty}\Gamma(s)\left\langle-A^{1/2}\eta'_s(s,\cdot)+A^{1/2}v(\cdot),A^{1/2}\eta(s,\cdot)\right\rangle_{\mathcal{H}}ds=$$

$$=-\frac{1}{2}\int_0^{+\infty}\Gamma(s)\frac{d}{ds}\left\|A^{1/2}\ \eta(s,\cdot)\right\|_{\mathcal{H}}^2ds=$$

$$=\frac{1}{2}\int_0^{+\infty}\Gamma'(s)\|\nabla\eta(s,\cdot)\|_{\mathcal{H}}^2ds\leq c\int_0^{+\infty}\Gamma(s)\|\nabla\ \eta(s,\cdot)\|_{\mathcal{H}}^2ds\leq c\|w\|_W^2.$$

Proof of Lemma 5. Suppose that $\mathcal{X}=(æ_1,æ_2,æ_3)\in W$. Consider the equation

$$\mathcal{A}z-\lambda z=\mathcal{X},w=(u,v,\eta)\in D(\mathcal{A}),$$

which is equivalent to the system of equations

$$\begin{cases} v-\lambda u=æ_1, \\ -[1+\ \int_0^{+\infty}\varGamma(s)ds)]\ Au+\int_0^{+\infty}\varGamma(s)A\eta(s)ds-\lambda v=æ_2, \\ -\eta_s+v-\lambda\eta=æ_3,\ \eta(0)=0. \end{cases} \tag{46}$$

Taking into account the second equation from the third equation of the system (46), we get that

$$\eta(\rho)=-\int_0^\rho e^{-\lambda(\rho-s)}æ_3(s)ds-\frac{1}{\lambda}(e^{\lambda\rho}-1)v.$$

Taking this into account, from the second equation of the system (46) we find that

$$\left[1+\ 2\int_0^{+\infty}\varGamma\,(s)ds\ -\int_0^{+\infty}\varGamma\,(s)e^{-\lambda s}ds\right]\ Au+\lambda^2 u=$$

$$=-\int_0^{+\infty}\varGamma(\rho)\int_0^\rho e^{-\lambda(\rho-s)}æ_3(s)dsd\rho-\lambda æ_1-æ_2-\frac{1}{\lambda}\int_0^{+\infty}\varGamma(s)(e^{-\lambda s}-1)dsAæ_1. \tag{47}$$

Since

$$1+\ 2\int_0^{+\infty}\varGamma\,(s)ds-\int_0^{+\infty}\varGamma\,(s)e^{-\lambda s}ds\geq 1+\ \int_0^{+\infty}\varGamma\,(s)ds>0,$$

the equation (47) can be written in the form

$$[A+K(\lambda)]w=G(\lambda),$$

where

$$K(\lambda)=\frac{\lambda^2}{1+\ 2\int_0^{+\infty}\varGamma\,(s)ds\ -\int_0^{+\infty}\varGamma\,(s)e^{-\lambda s}ds},$$

$$G(\lambda)=\frac{-\int_0^{+\infty}\varGamma\,(\rho)\int_0^\rho e^{-\lambda(\rho-s)}æ_3\,(s)\,dsd\rho-\lambda æ_1-æ_2-\frac{1}{\lambda}\int_0^{+\infty}\varGamma(s)(e^{-\lambda s}-1)ds\ Aæ_1}{1+2\int_0^{+\infty}\varGamma(s)ds-\int_0^{+\infty}\varGamma(s)e^{-\lambda s}ds}.$$

Since A is a self-adjoint positive operator and $K(\lambda)>0$ therefore $A+K(\lambda)$ is an invertible operator, and $w=[A+K(\lambda)]^{-1}G(\lambda)$.

References

1. Aliev A.B., Mammadhasanov E.H. Well-posedness of initial boundary value problem on longitudinal impact on a composite linear viscoelastic bar. *Math. Methods Appl. Sci.*, 2017, **40** (14), pp. 5380–5390.
2. Arenz R.J. Uniaxial wave propagation in realistic viscoelastic materials. *J. Appl. Mech.*, 1964, **31** (1), pp. 17–21.
3. Berry D.S., Hunter S.C. The propagation of dynamic stresses in visco-elastic rods. *J. Mech. Phys. Solids*, 1956, **4** (2), pp. 72–95.

4. Chu B.T. Stress wave in isotropic linear viscoelastic materials. *J. Mécanique*, 1962, **1** (1), pp. 439–462.
5. Dafermos C.M. Asymptotic stability in viscoelasticity. *Arch. Rational Mech. Anal.*, 1970, **37**, pp. 297–308.
6. Il'yasov M.Kh., Mammadhasanov E.H. Distribution of non-stationary waves in the composite viscous elastic rod under proportional creep kernels. *Izv. Akad. Nauk Azerb. SSR, Ser. Fiz.-Tekh. Mat. Nauk*, 1989, **10** (6), pp. 54–60 (in Russian).
7. Kato T. *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1995.
8. Lee E.H., Kanter I. Wave propagation in finite rods of viscoelastic material. *J. Appl. Phys.*, 1953, **24**, pp. 1115–1122.
9. Lee T.M. Spherical waves in viscoelastic media. *J. Acoustical Soc. America*, 1964, **36** (12), pp. 2402–2407.
10. Lions J.L., Magenes E. *Non-Homogeneous Boundary Value Problems and Applications*. Dunod, Paris, 1968; Mir, Moscow, 1971; Springer, Berlin, 1972.
11. Lokshin A.A., Suvorova Y.V. *Mathematical Theory of Wave Propagation in Media with Memory*. Moscow University, Moscow, 1982 (in Russian).
12. Maksudov F.G., Il'yasov M.Kh. On a method for solving dynamic problems of linear viscoelasticity with regular hereditary kernels.*Dokl. Akad. Nauk SSSR*, 1981, **260** (6), pp. 1332–1335 (in Russian).
13. Sabodash P. F. Propagation of longitudinal viscoelastic waves in a three-layer medium. *Polymer Mechanics*, 1971, **7** (1), pp.124–128.
14. Zverev I.N. The propagation of a disturbance in a visco-elastic and visco-plastic bar. *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 1950, **14** (3), pp. 295–302 (in Russian).