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CORRECT PROOF OF FINDING THE EXACT LOWER BOUND OF THE RAYLEIGH MAGNETIC VALUE

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Abstract. *In the two-dimensional space* $\mathbb{R}^2$*, we consider the magnetic relation of Rayleigh:*

$$\frac{\int_{\mathbb{R}^2} \left|(i\nabla+\omega)\,\psi(x)\right|^2 dx}{\int_{\mathbb{R}^2} \left|\psi(x)\right|^2 dx},$$

where $\omega(x)=\frac{1}{2}(-x_2,x_1)$*, which appears in the mathematical theory of surface superconductivity when studying the first eigenvalue of the Landau operator. In a simple way, it is proved that the exact lower bound in the first-order Sobolev space,* $H^1(\mathbb{R}^2)$*, of the Rayleigh magnetic ratio is equal to one.*

Keywords: magnetic fields, superconductivity, Ginsburg-Landau operator, Rayleigh quotient, eigenvalue

Mathematics Subject Classification (2020): 35J20, 35Q56, 35Q40, 81Q10

1. Introduction

When studying surface superconductivity in a type II superconducting material with different cross sections, it becomes necessary (see [1]-[11]), under the growth of the intensity of an external magnetic field, to study the asymptotic behavior of the first eigenvalue and the corresponding eigenfunction of the system of Ginsburg-Landau equations (see [3, p. 143])

$$\begin{cases} \left.\begin{array}{r} (i\nabla+hA)^2\,\psi(x)=\frac{h^2}{\sigma^2}\left(1-|\psi(x)|^2\right)\psi(x)\,, \\ curl\,(curlA-\beta)=-\frac{1}{h}Re\left[\overline{\psi(x)}\,(i\nabla+hA)\,\psi(x)\right] \end{array}\right\} \text{ in } \Omega\,, \\ \left.\begin{array}{r} (i\nabla+hA)\,\psi(x)\cdot\nu=0\,, \\ curlA=\beta \end{array}\right\} \text{ on } \partial\Omega\,, \end{cases}$$

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where $x = (x_1, x_2) \in \mathbb{R}^2$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, β is the external magnetic field, h is the magnetic field strength, $A = (a_1(x_1, x_2), a_2(x_1, x_2))$ is the induced real magnetic potential, $i = \sqrt{-1}$, $\sigma > 0$ is the Ginsburg-Landau parameter, Ω is the material cross section, $\partial\Omega$ is the boundary of the domain Ω, $\psi(x)$ is the wave function, ν is the external normal vector, and

$$curlA = \frac{\partial a_2(x_1, x_2)}{\partial x_1} - \frac{\partial a_1(x_1, x_2)}{\partial x_2}, curl^2 A = \left(\frac{\partial(curlA)}{\partial x_2}, -\frac{\partial(curlA)}{\partial x_1}\right).$$

From the second variation (see [10]) of the Ginsburg-Landau functional

$$G(\psi, A) = \int_\Omega \left\{ |(i\nabla + hA)\psi(x)|^2 + \frac{h^2}{\sigma^2}\left(|\psi(x)|^2 - 1\right)^2 \right\} dx +$$

$$+h^2 \int_{\mathbb{R}^2} |curlA - \beta|^2 dx$$

in the vicinity of the normal state $\psi(x) = 0$ it can be seen that this problem is closely related to finding the exact lower bound of the Rayleigh magnetic quantity

$$\frac{\int_{\mathbb{R}^2} |(i\nabla + hA)\psi(x)|^2 dx}{\int_{\mathbb{R}^2} |\psi(x)|^2 dx} \tag{1}$$

in the first-order Sobolev space, $W_2^1(\mathbb{R}^2) = \overset{0}{W_2^1}(\mathbb{R}^2) = H^1(\mathbb{R}^2)$.

Due to the gauge invariance of expression (1) (see [3], [8], [10]) and equality (see [8, Lemma 2.1])

$$\alpha(h) = \inf_{\psi(x) \in H^1(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |(i\nabla + h\omega(x))\psi(x)|^2 dx}{\int_{\mathbb{R}^2} |\psi(x)|^2 dx} = \alpha(1)|h|, \tag{2}$$

where $\omega(x) = \frac{1}{2}(-x_2, x_1)$, it suffices to find an explicit value of $\alpha(1)$.

The following theorem holds.

Theorem 1. *[see [8, Theorem 2.2], [10, Proposition 2.7]] Let $A = \omega(x)$. Then $\alpha(1) = 1$.*

Before proceeding to the proof of the theorem, we note some remarks.

1°. As noted in [10], this result is well known from the physical literatures and in some form goes back to Landau.

2°. In [8], in contrast to [10], where the infimum is taken over the set $H^1(\mathbb{R}^2)$, in formula (2) the infimum is taken over the set $W(\mathbb{R}^2) = W^{1,2}_{loc}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. But it does not change $\alpha(h)$ (including $\alpha(1)$).

3°. In both works, calculating the Rayleigh value

$$\frac{\int_{\mathbb{R}^2} |(i\nabla + \omega(x))\psi(x)|^2 dx}{\int_{\mathbb{R}^2} |\psi(x)|^2 dx}$$

for the function $\psi(x) = e^{-\frac{r^2}{4}}$, where $r = \sqrt{x_1^2 + x_2^2}$, it is established that $\alpha(1) \le 1$. But when proving the inequality $\alpha(1) \ge 1$, various arguments are applied.

4°. In [10], it is noted that unfortunately in [8], in the proof of the inequality $\alpha(1) \ge 1$, numerous mistakes were made and a new proof of this inequality is given. But we note that unfortunately, in [10] also, mistakes are made in the proof of the inequality $\alpha(1) \ge 1$. So, in [10], they introduce the sets

$$A_k = \left\{ u(r) \in C_0^1([0, +\infty)\ ;\ \mathbb{R}) : \quad u(0) = 0,\ if\ k \in \mathbb{Z} \setminus \{0\} \right\},$$

where $\mathbb{Z}$ is the set of integers, and it is claimed that

$$J_k(u_k) \ge \inf_{A_k} \left\{ \frac{\int_0^{+\infty} \left(|u'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |u(r)|^2 \right) rdr}{\int_0^{+\infty} |u(r)|^2 rdr} \right\},\ k \in \mathbb{Z} \setminus \{0\}, \tag{3}$$

where

$$J_k(u_k) = \frac{\int_0^{+\infty} \left(|u_k'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |u_k(r)|^2 \right) rdr}{\int_0^{+\infty} |u_k(r)|^2 rdr}, \quad k \in \mathbb{Z},$$

$$u_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\psi}(r, \theta) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z}, \tag{4}$$

$$\tilde{\psi}(r, \theta) = \psi(r\cos\theta, r\sin\theta) = \sum_{k=-\infty}^{+\infty} u_k(r) e^{ik\theta}, \tag{5}$$

$$\psi(x_1, x_2) \in C_0^\infty\left(\mathbb{R}^2\right).$$

Then using the equality

$$\int_{\mathbb{R}^2} |(i\nabla + A)\psi(x)|^2 dx = 2\pi \sum_{k=-\infty}^{+\infty} J_k(u_k) \int_0^{+\infty} |u_k(r)|^2 rdr \tag{6}$$

and the inequalities

$$\inf_{A_k} \left\{ \frac{\int_0^{+\infty} \left(|u'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |u(r)|^2 \right) rdr}{\int_0^{+\infty} |u(r)|^2 rdr} \right\} \ge 1$$

for $k \in \mathbb{Z} \setminus \{0\}$, it is claimed that

$$\int_{\mathbb{R}^2} |(i\nabla + A)\psi(x)|^2 dx \ge \int_{\mathbb{R}^2} |\psi(x)|^2 dx.$$

There are two flaws in this reasoning. First, in the equality (6) there is $J_0(u_0)$. It should have been estimated from below. Secondly, the set A_k $(k \in \mathbb{Z} \setminus \{0\})$ includes real-valued functions, and the functions $u_k(r)$ $(k \in \mathbb{Z})$ are complex-valued functions. Therefore, generally speaking, $u_k(r) \notin A_k$. If there would be a real-valued function $h(r)$ from the set A_k such that

$$J_k(u_k) = J_k(h) = \frac{\int_0^{+\infty} \left(|h'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |h(r)|^2\right) rdr}{\int_0^{+\infty} |h(r)|^2 rdr}, \tag{7}$$

then statement (3) would be true. It is easy to verify that the complex-valued function $u_k(r)$, which satisfies equality (7), should have the following form: $u_k(r) = (c_1 + ic_2)\varphi_k(r)$, where $\varphi_k(r) \in A_k$, c_1 and c_2 are real numbers. But, generally speaking, the function $u_k(r)$ does not have to be in the form of $u_k(r) = (c_1 + ic_2)\varphi_k(r)$.

5°. In both papers, they use the unobvious equality

$$\int_{\mathbb{R}^2} |(i\nabla + A)\psi(x)|^2 dx =$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} \left(|u_k'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |u_k(r)|^2\right) rdr, \tag{8}$$

where $\psi(x) \in C_0^\infty(\mathbb{R}^2)$ and $u_k(r)$ are determined by the formula (4).

2. Correct Proof of Theorem 1

In view of the non-obviousness of equality (8), we present its proof.

Lemma 1. *Let $\psi(x) \in C_0^\infty(\mathbb{R}^2)$. Then equality (8) is true.*

Proof. Let $\psi(x) \in C_0^\infty(\mathbb{R}^2)$. Given the apparent form of the magnetic potential, $\omega(x) = \frac{1}{2}(-x_2, x_1)$, we have:

$$\int_{\mathbb{R}^2} |(i\nabla + A)\psi(x)|^2 dx = \int_{\mathbb{R}^2} \left(\left|i\frac{\partial\psi}{\partial x_1} - \frac{x_2}{2}\psi\right|^2 + \left|i\frac{\partial\psi}{\partial x_2} + \frac{x_1}{2}\psi\right|^2\right) dx \equiv$$

$$\equiv \int_{\mathbb{R}^2} \left(|h(x_1, x_2)|^2 + |g(x_1, x_2)|^2\right) dx, \tag{9}$$

where

$$h(x_1, x_2) = i\frac{\partial\psi}{\partial x_1} - \frac{x_2}{2}\psi, g(x_1, x_2) = i\frac{\partial\psi}{\partial x_2} + \frac{x_1}{2}\psi.$$

Moving to the polar coordinate system, we obtain:

$$\begin{aligned} \tilde{h}(r,\theta) &= h(r\cos\theta, r\sin\theta) = i\left[\frac{\partial\tilde{\psi}}{\partial r}\cos\theta - \frac{\partial\tilde{\psi}}{\partial\theta}\frac{\sin\theta}{r}\right] - \frac{r\sin\theta}{2}\tilde{\psi}, \\ \tilde{g}(r,\theta) &= g(r\cos\theta, r\sin\theta) = i\left[\frac{\partial\tilde{\psi}}{\partial r}\sin\theta + \frac{\partial\tilde{\psi}}{\partial\theta}\frac{\cos\theta}{r}\right] + \frac{r\cos\theta}{2}\tilde{\psi}, \end{aligned} \tag{10}$$

where $\tilde{\psi} = \tilde{\psi}(r,\theta) = \psi(r\cos\theta, r\sin\theta)$.

Given the expansion

$$\tilde{h}(r,\theta) = \sum_{k=-\infty}^{+\infty} h_k(r)\, e^{ik\theta} \text{ and } \tilde{g}(r,\theta) = \sum_{k=-\infty}^{+\infty} g_k(r)\, e^{ik\theta},$$

where

$$h_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{h}(r,\theta)\, e^{-ik\theta} d\theta, \quad g_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{g}(r,\theta)\, e^{-ik\theta} d\theta, \quad k \in \mathbb{Z},$$

and Parseval's equality, we rewrite (9) in the following form:

$$\int_{\mathbb{R}^2} |(i\nabla + \omega)\,\psi(x)|^2\, dx = 2\pi \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} \left(|h_k(r)|^2 + |g_k(r)|^2\right) r dr. \tag{11}$$

Using equation (10) and the expansion (5), we calculate $h_k(r)$ and $g_k(r)$ ($k \in \mathbb{Z}$).
We have:

$$h_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ i\left[\frac{\partial\tilde{\psi}}{\partial r}\cos\theta - \frac{\partial\tilde{\psi}}{\partial\theta}\frac{\sin\theta}{r}\right] - \frac{r\sin\theta}{2}\tilde{\psi}\right\} e^{-ik\theta} d\theta =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} i\frac{\partial\tilde{\psi}}{\partial r}\frac{e^{-i(k-1)\theta} + e^{-i(k+1)\theta}}{2} d\theta - \frac{i}{2\pi}\frac{1}{r}\int_{-\pi}^{\pi}\frac{\partial\tilde{\psi}}{\partial\theta}\frac{e^{-i(k-1)\theta} - e^{-i(k+1)\theta}}{2i} d\theta -$$

$$-\frac{1}{2\pi}\frac{r}{2}\int_{-\pi}^{\pi}\tilde{\psi}\frac{e^{-i(k-1)\theta} - e^{-i(k+1)\theta}}{2i} d\theta = \frac{i}{2}\left(u'_{k-1} + u'_{k+1}\right) -$$

$$-\frac{1}{2\pi}\frac{1}{2r}\int_{-\pi}^{\pi}\frac{\partial\tilde{\psi}}{\partial\theta}\left(e^{-i(k-1)\theta} - e^{-i(k+1)\theta}\right) d\theta + \frac{r}{4} i\left(u_{k-1} + u_{k+1}\right). \tag{12}$$

Now we calculate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\tilde{\psi}}{\partial\theta}\left(e^{-i(k-1)\theta} - e^{-i(k+1)\theta}\right) d\theta.$$

Integrating by parts and taking into account the equality

$$e^{-i(k\pm1)\theta}\tilde{\psi}(r,\theta)\Big|_{-\pi}^{\pi} = 2i\tilde{\psi}(-r,0)\sin(k\pm1)\pi = 0,$$

we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\tilde{\psi}}{\partial\theta}\left(e^{-i(k-1)\theta} - e^{-i(k+1)\theta}\right) d\theta = i(k-1)\,u_{k-1} - i(k+1)\,u_{k+1}. \tag{13}$$

Given (13) in (12), we obtain

$$h_k(r) = \frac{i}{2}\left\{\left(u'_{k-1} + u'_{k+1}\right) -\right.$$

$$-\left(\frac{k-1}{r}-\frac{r}{2}\right)u_{k-1}+\left(\frac{k+1}{r}-\frac{r}{2}\right)u_{k+1}\Big\},\ k\in\mathbb{Z}. \quad (14)$$

By doing the same for $g_k(r)$ $(k\in\mathbb{Z})$, we obtain the following formula:

$$g_k(r)=\frac{1}{2}\left\{\left(u'_{k-1}-u'_{k+1}\right)-\right.$$

$$\left.-\left(\frac{k-1}{r}-\frac{r}{2}\right)u_{k-1}-\left(\frac{k+1}{r}-\frac{r}{2}\right)u_{k+1}\right\},\ k\in\mathbb{Z}. \quad (15)$$

Considering formulas (14) and (15) in equality (11), we obtain

$$\int_{\mathbb{R}^2}\left|(i\nabla+\omega)\psi(x)\right|^2dx=2\pi\int_0^{+\infty}\frac{1}{4}\left\{\sum_{k=-\infty}^{+\infty}\left[\left|\left(u'_{k-1}+u'_{k+1}\right)\right.\right.\right.-$$

$$\left.-\left(\frac{k-1}{r}-\frac{r}{2}\right)u_{k-1}+\left(\frac{k+1}{r}-\frac{r}{2}\right)u_{k+1}\right|^2+$$

$$\left.\left.+\left|\left(u'_{k-1}-u'_{k+1}\right)-\left(\frac{k-1}{r}-\frac{r}{2}\right)u_{k-1}-\left(\frac{k+1}{r}-\frac{r}{2}\right)u_{k+1}\right|^2\right]\right\}rdr. \quad (16)$$

Let $\sigma(x_1,x_2)=Re\psi(x_1,x_2)$ and $\tau(x_1,x_2)=Im\psi(x_1,x_2)$. Then for any integer k we have:

$$u_k(r)=\sigma_k(r)+i\tau_k(r)=u_k(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\tilde{\sigma}(r,\theta)+i\tilde{\tau}(r,\theta)\right]e^{-ik\theta}d\theta, \quad (17)$$

where $\tilde{\sigma}(r,\theta)=\sigma(r\cos\theta,r\sin\theta)$ and $\tilde{\tau}(r,\theta)=\tau(r\cos\theta,r\sin\theta)$.

Given the representation (17) in the formula (16), we obtain:

$$\int_{\mathbb{R}^2}\left|(i\nabla+\omega)\psi(x)\right|^2dx=$$

$$=2\pi\int_0^{+\infty}\frac{1}{4}\left\{\sum_{k=-\infty}^{+\infty}\left\{\left[\left(\sigma'_{k-1}+\sigma'_{k+1}\right)\right.\right.\right.-\left(\frac{k-1}{r}-\frac{r}{2}\right)\sigma_{k-1}+\left.\left(\frac{k+1}{r}-\frac{r}{2}\right)\sigma_{k+1}\right]^2+$$

$$+\left[\left(\sigma'_{k-1}-\sigma'_{k+1}\right)-\left(\frac{k-1}{r}-\frac{r}{2}\right)\sigma_{k-1}-\left(\frac{k+1}{r}-\frac{r}{2}\right)\sigma_{k+1}\right]^2+$$

$$+\left[\left(\tau'_{k-1}+\tau'_{k+1}\right)-\left(\frac{k-1}{r}-\frac{r}{2}\right)\tau_{k-1}+\left(\frac{k+1}{r}-\frac{r}{2}\right)\tau_{k+1}\right]^2+$$

$$\left.\left.\left.+\left[\left(\tau'_{k-1}-\tau'_{k+1}\right)-\left(\frac{k-1}{r}-\frac{r}{2}\right)\tau_{k-1}-\left(\frac{k+1}{r}-\frac{r}{2}\right)\tau_{k+1}\right]^2\right\}\right\}rdr=$$

$$=2\pi\int_0^{+\infty}\frac{1}{4}\left\{\sum_{k=-\infty}^{+\infty}\left[2\left(\sigma'_{k-1}\right)^2+2\left(\sigma'_{k+1}\right)^2+2\left(\frac{k+1}{r}-\frac{r}{2}\right)^2\left(\sigma_{k+1}\right)^2+\right.\right.$$

$$+2\left(\frac{k-1}{r}-\frac{r}{2}\right)^2(\sigma_{k-1})^2+2\left(\sigma'_{k-1}+\sigma'_{k+1}\right)\left(\frac{k+1}{r}-\frac{r}{2}\right)\sigma_{k+1}-$$

$$-2\left(\sigma'_{k-1}+\sigma'_{k+1}\right)\left(\frac{k-1}{r}-\frac{r}{2}\right)\sigma_{k-1}-2\left(\sigma'_{k-1}-\sigma'_{k+1}\right)\left(\frac{k-1}{r}-\frac{r}{2}\right)\sigma_{k-1}-$$

$$-\ 2\left(\sigma'_{k-1}-\sigma'_{k+1}\right)\left(\frac{k+1}{r}-\frac{r}{2}\right)\sigma_{k+1}\Bigg]\Bigg\}rdr+$$

$$+2\pi\int_0^{+\infty}\frac{1}{4}\left\{\sum_{k=-\infty}^{+\infty}\left[2\left(\tau'_{k-1}\right)^2+2\left(\tau'_{k+1}\right)^2+2\left(\frac{k+1}{r}-\frac{r}{2}\right)^2(\tau_{k+1})^2\ +\right.\right.$$

$$+2\left(\frac{k-1}{r}-\frac{r}{2}\right)^2(\tau_{k-1})^2+2\left(\tau'_{k-1}+\tau'_{k+1}\right)\left(\frac{k+1}{r}-\frac{r}{2}\right)\tau_{k+1}-$$

$$-2\left(\tau'_{k-1}+\tau'_{k+1}\right)\left(\frac{k-1}{r}-\frac{r}{2}\right)\tau_{k-1}-2\left(\tau'_{k-1}-\tau'_{k+1}\right)\left(\frac{k-1}{r}-\frac{r}{2}\right)\tau_{k-1}-$$

$$-\ 2\left(\tau'_{k-1}-\tau'_{k+1}\right)\left(\frac{k+1}{r}-\frac{r}{2}\right)\tau_{k+1}\Bigg]\Bigg\}rdr=$$

$$=2\pi\int_0^{+\infty}\left\{\sum_{k=-\infty}^{\infty}\left[(\sigma'_k)^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2\sigma_k^2\right]\right\}rdr+$$

$$+2\pi\int_0^{+\infty}\left\{\sum_{k=-\infty}^{\infty}\left[(\tau'_k)^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2\tau_k^2\right]\right\}rdr=$$

$$=2\pi\sum_{k=-\infty}^{\infty}\int_0^{+\infty}\left(|u'_k|^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2|u_k|^2\right)\ rdr.$$

The lemma is proved. ◄

Proof of the Theorem 1. Let $\psi(x_1,x_2)\in C_0^\infty\left(\mathbb{R}^2\right)$ and $u_k(r)$ $(k\in\mathbb{Z})$ is determined by the formula (4). Let us show that for any integer k the following inequality is true:

$$J_k(u_k)=\frac{\int_0^{+\infty}\left(|u'_k(r)|^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2|u_k(r)|^2\right)\ rdr}{\int_0^{+\infty}|u_k(r)|^2\,rdr}\geq 1. \tag{18}$$

Step I. $k\in\mathbb{Z}\backslash\{0\}$. Let's introduce the notation:

$$E=\left\{u(r)\in C^\infty([0,+\infty);\ \mathbb{C}):\ u(0)=0,\ \exists r_u>0,\ supp\,u(r)\subset[0,r_u]\right\},$$

where $supp\,u(r)$ is the carrier of the function $u(r)$. Obviously, if $\psi(x_1,x_2)\in C_0^\infty\left(\mathbb{R}^2\right)$, then for any $k\in\mathbb{Z}\backslash\{0\}$ $u_k(r)\in E$. Let $\sigma(r)+i\tau(r)=u(r)\in E$. Using the fact that if $u(r)\in E$, then the functions $\sigma(r)=Re\,u(r)$ and $\tau(r)=Im\,u(r)$ also belong to the set E, we obtain:

$$\int_0^{+\infty}\left(|u'(r)|^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2|u(r)|^2\right)\ rdr=\int_0^{+\infty}\left[(\sigma'(r))^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2\sigma^2(r)\right]rdr+$$

$$+\int_0^{+\infty}\left[\left(\tau'(r)\right)^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2\tau^2(r)\right]rdr\geq\int_0^{+\infty}2\sigma(r)\,\sigma'(r)\left(\frac{k}{r}-\frac{r}{2}\right)rdr+$$

$$+\int_0^{+\infty}2\tau(r)\,\tau'(r)\left(\frac{k}{r}-\frac{r}{2}\right)rdr=\int_0^{+\infty}\left[\sigma^2(r)\right]'\left(k-\frac{r^2}{2}\right)dr+$$

$$+\int_0^{+\infty}\left[\tau^2(r)\right]'\left(k-\frac{r^2}{2}\right)dr=\sigma^2(r)\left(k-\frac{r^2}{2}\right)\Big|_0^{+\infty}+$$

$$+\int_0^{+\infty}\sigma^2(r)\,rdr+\tau^2(r)\left(k-\frac{r^2}{2}\right)\Big|_0^{+\infty}+\int_0^{+\infty}\tau^2(r)\,rdr=$$

$$=\int_0^{+\infty}\left[\sigma^2(r)+\tau^2(r)\right]rdr=\int_0^{+\infty}|u(r)|^2\,rdr.$$

Consequently, for any function from the set E, the following inequality is true

$$\int_0^{+\infty}\left(|u'(r)|^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2|u(r)|^2\right)rdr\geq\int_0^{+\infty}|u(r)|^2\,rdr. \tag{19}$$

From inequality (19), in particular, we obtain

$$J_k(u_k)=\frac{\int_0^{+\infty}\left(|u_k'(r)|^2+\left(\frac{k}{r}-\frac{r}{2}\right)^2|u_k(r)|^2\right)rdr}{\int_0^{+\infty}|u_k(r)|^2\,rdr}\geq 1,\quad k\in\mathbb{Z}\setminus\{0\}. \tag{20}$$

Step II. $k=0$. Let's introduce the set

$$L=\left\{u(r)\in C^{\infty}([0,+\infty);\,\mathbb{C}):\ \exists r_u>0,\ supp\,u(r)\subset[0,r_u]\right\}.$$

Obviously, the functions

$$u_0(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\psi}(r,\theta)\,d\theta,\ \ \sigma_0(r)=Re u_0(r)\ \text{ and }\ \tau_0(r)=Im u_0(r)$$

are elements of the set L. Using these facts, to estimate $J_0(u_0)$ from below, we perform the following calculations:

$$\int_0^{+\infty}\left(|u_0'(r)|^2+\left(-\frac{r}{2}\right)^2|u_0(r)|^2\right)rdr=\int_0^{+\infty}\left[\left(\sigma_0'(r)\right)^2+\left(-\frac{r}{2}\right)^2\sigma_0^2(r)\right]rdr+$$

$$+\int_0^{+\infty}\left[\left(\tau_0'(r)\right)^2+\left(-\frac{r}{2}\right)^2\tau_0^2(r)\right]rdr\geq\int_0^{+\infty}2\sigma_0(r)\,\sigma_0'(r)\left(-\frac{r}{2}\right)rdr+$$

$$+\int_0^{+\infty}2\tau_0(r)\,\tau_0'(r)\left(-\frac{r}{2}\right)rdr=$$

$$=\int_0^{+\infty}\left(\sigma_0^2(r)\right)'\left(-\frac{r^2}{2}\right)dr+\int_0^{+\infty}\left(\tau_0^2(r)\right)'\left(-\frac{r^2}{2}\right)dr=$$

$$= \sigma_0^2(r)\left(-\frac{r^2}{2}\right)\Big|_0^{+\infty} + \int_0^{+\infty} \sigma_0^2(r)\, rdr + \tau_0^2(r)\left(-\frac{r^2}{2}\right)\Big|_0^{+\infty} +$$

$$+\int_0^{+\infty} \tau_0^2(r)\, rdr = \int_0^{+\infty} \left[\sigma_0^2(r) + \tau_0^2(r)\right] rdr = \int_0^{+\infty} |u_0(r)|^2\, rdr.$$

Therefore,

$$\int_0^{+\infty} \left(|u_0'(r)|^2 + \left(-\frac{r}{2}\right)^2 |u_0(r)|^2\right) rdr \geq \int_0^{+\infty} |u_0(r)|^2\, rdr.$$

Thus, we have come to the inequality

$$J_0(u_0) = \frac{\int_0^{+\infty} \left(|u_0'(r)|^2 + \left(-\frac{r}{2}\right)^2 |u_0(r)|^2\right) rdr}{\int_0^{+\infty} |u_0(r)|^2\, rdr} \geq 1. \tag{21}$$

From inequalities (20) and (21), we have that for any integer k, inequality (18) holds. From equality (8) and inequality (18), we obtain that for any function from the space $C_0^\infty(\mathbb{R}^2)$, the following inequality holds:

$$\int_{\mathbb{R}^2} |(i\nabla + \omega)\psi(x)|^2\, dx =$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} \left\{ \frac{\int_0^{+\infty} \left(|u_k'(r)|^2 + \left(\frac{k}{r} - \frac{r}{2}\right)^2 |u_k(r)|^2\right) rdr}{\int_0^{+\infty} |u_k(r)|^2\, rdr} \cdot \int_0^{+\infty} |u_k(r)|^2\, rdr \right\} =$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} \left[J_k(u_k) \cdot \int_0^{+\infty} |u_k(r)|^2\, rdr\right] \geq$$

$$\geq 2\pi \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} |u_k(r)|^2\, rdr = \int_{\mathbb{R}^2} |\psi(x)|^2\, dx. \tag{22}$$

Since the space of basic functions $C_0^\infty(\mathbb{R}^2)$ is everywhere dense in the space $H^1(\mathbb{R}^2)$, it follows from inequality (22) that for any function from the space $H^1(\mathbb{R}^2)$, the following inequality is true

$$\frac{\int_{\mathbb{R}^2} |(i\nabla + \omega)\psi(x)|^2\, dx}{\int_{\mathbb{R}^2} |\psi(x)|^2\, dx} \geq 1.$$

From here we have

$$\alpha(1) = \inf_{H^1(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |(i\nabla + \omega)\psi(x)|^2\, dx}{\int_{\mathbb{R}^2} |\psi(x)|^2\, dx} \geq 1.$$

The theorem is proved. ◀

References

1. Dauge M., Miqueu J.-P., Raymond N. On the semiclassical Laplacian with magnetic field having self-intersecting zero set, *arXiv:1807.09028 [math.AP]*.
2. Dombrowski N., Raymond N. Semiclassical analysis with vanishing magnetic fields. *J. Spectr. Theory*, 2013, **3** (3), pp. 423–464.
3. Fournais S., Helffer B. *Spectral methods in surface superconductivity.* Progress in Nonlinear Differential Equations and their Applications, 77. Birkhäuser Boston, Inc., Boston, MA, 2010.
4. Ginzburg V.L., Landau L.D. On the theory of superconductivity. *Zh. Eksper. Teoret. Fiz.*, 1950, **20**, pp.1064–1082 (English transl., Collected Papers of L. D. Landau, Gordon & Breach, New York, 1967, pp. 546–568).
5. Helffer B., Mohamed A. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Func. Anal.*, 1996, **138** (1), pp. 40–81.
6. Helffer B., Morame A. Magnetic bottles in connection with superconductivity. *J. Func. Anal.*, 2001, **185** (2), pp. 604–680.
7. Lu K., Pan X.-B. Eigenvalue problems for Ginzburg-Landau operator in bounded domains. *J. Math. Phys.*, 1999, **40** (6), pp. 2647–2670.
8. Lu K., Pan X.-B. Gauge invariant eigenvalue problems in $\mathbb{R}^2$ and $\mathbb{R}^2_+$. *Trans. Amer. Math. Soc.*, 2000, **352** (3), pp. 1247–1276.
9. Miqueu J.-P. Eigenstates of the Neumann magnetic Laplacian with vanishing magnetic field. *Ann. Henri Poincaré*, 2018, **19** (7), pp. 2021–2068.
10. del Pino M., Felmer P.L., Sternberg P. Boundary concentration for eigenvalue problems related to the onset of superconductivity. *Comm. Math. Phys.*, 2008, **210** (2), pp. 413–446.
11. Raymond N., Vũ Ngọc S. Geometry and spectrum in 2D magnetic wells. *Ann. Inst. Fourier (Grenoble)*, 2015, **65** (1), pp. 137–169.