UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR DIFFERENTIAL OPERATOR WITH SEMISEPARATED BOUNDARY CONDITIONS

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Abstract. In the article we consider the Sturm-Liouville operator with semiseparated boundary conditions, one of which contains a spectral parameter. An asymptotic formula for the eigenvalues of the operator under consideration is given and a uniqueness theorem for the solution of the inverse problem of recovering the corresponding boundary value problems is proved.

Keywords: Sturm-Liouville operator, eigenvalues, inverse problem, uniqueness theorem

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1. Introduction

Boundary-value problems with boundary conditions depending on the spectral parameter often arise in various fields of natural science and technology in the study of a number of problems, the construction of systems for the protection of devices against impact, vibrations of a string with a load at the end, torsional vibrations of a shaft with a flywheel at the end, vibrations of antennas loaded with concentrated capacities and inductances, etc. (see, for example, [1], [11] and the literature there). Inverse spectral problems associated with problems of this type also play an important role in the study of some nonlinear evolutionary equations of mathematical physics.

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Consider a boundary value problem generated on an interval \([0, \pi]\) by the Sturm-Liouville equation
\[
- y''(x) + q(x)y(x) = \lambda^2 y(x) \tag{1}
\]
and semiseparated boundary conditions of the form
\[
\begin{align*}
y'(0) + \alpha y(0) &= 0, \\
y(0) + \lambda [\beta y(\pi) + \gamma y'(\pi)] &= 0,
\end{align*}
\tag{2}
\]
where \(q(x)\) is a real function belonging to the space \(L_2[0, \pi]\), \(\lambda\) is a spectral parameter, \(\alpha, \beta, \gamma\) are real numbers. This problem will be denoted by \(P(\alpha, \beta, \gamma)\). In this paper, we present an asymptotic formula for the eigenvalues of the considered Sturm-Liouville operator and prove a uniqueness theorem for the solution of the inverse problem of recovering the corresponding boundary value problems from spectral data. The spectra of two boundary value problems and a certain number are used as spectral data. Note that earlier the question of recovering differential operators for separated and nonseparated boundary conditions containing a spectral parameter was studied in [2], [4]-[7], [11]-[14] and other works.

2. Spectral Data of Boundary Value Problems

We denote by \(c(x, \lambda), s(x, \lambda)\), solutions of equation (1), satisfying the initial conditions
\[
c(0, \lambda) = s'(0, \lambda) = 1, c'(0, \lambda) = s(0, \lambda) = 0.
\]
For any \(x\) function \(c(x, \lambda), s(x, \lambda), c'(x, \lambda), s'(x, \lambda)\) are entire functions (of exponential type) of variable \(\lambda\). The general solution of equation (1) is written in the form
\[
y(x, \lambda) = A_1 c(x, \lambda) + A_2 s(x, \lambda),
\]
where \(A_1, A_2\) are arbitrary constants. Substituting this function into the boundary conditions (2) and using the last relations, we obtain for \(A_1\) and \(A_2\) the following system:
\[
\begin{align*}
A_1 + \alpha A_2 &= 0, \\
[\lambda \beta s(\pi, \lambda) + \lambda \gamma s'(\pi, \lambda)] A_1 + [1 + \lambda \beta c(\pi, \lambda) + \lambda \gamma c'(\pi, \lambda)] A_2 &= 0.
\end{align*}
\]
For a number \(\lambda\) to be an eigenvalue of a boundary value problem \(P(\alpha, \beta, \gamma)\), it is necessary and sufficient that the latter system has a nonzero solution. But this system has a nonzero solution if and only if its determinant is equal to zero. Therefore, the eigenvalues of the boundary value problem \(P(\alpha, \beta, \gamma)\) coincide with the roots of the equation \(\Delta(\lambda) = 0\), where
\[
\Delta(\lambda) = \left| \lambda \beta s(\pi, \lambda) + \lambda \gamma s'(\pi, \lambda) \right| = \left| 1 + \lambda \beta c(\pi, \lambda) + \lambda \gamma c'(\pi, \lambda) \right|.
\]
This function is called the characteristic function of the problem \(P(\alpha, \beta, \gamma)\). Expanding the last determinant, we have
\[
\Delta(\lambda) = 1 + \lambda [\beta(c(\pi, \lambda) - \alpha s(\pi, \lambda)) + \gamma (c'(\pi, \lambda) - \alpha s'(\pi, \lambda))]. \tag{3}
\]
Since $\Delta(0) = 1$, that $\lambda = 0$ is not an eigenvalue of the boundary value problem $P(\alpha, \beta, \gamma)$.

It is known [10, p. 47] that the functions $c(\pi, \lambda), c'(\pi, \lambda), s(\pi, \lambda)$ and $s'(\pi, \lambda)$ hold the following representations:

$$c(\pi, \lambda) = \cos \lambda \pi + Q \frac{\sin \lambda \pi}{\lambda} + \frac{f_1(\lambda)}{\lambda},$$

$$c'(\pi, \lambda) = -\lambda \sin \lambda \pi + Q \cos \lambda \pi + f_2(\lambda),$$

$$s(\pi, \lambda) = \sin \lambda \pi - Q \frac{\cos \lambda \pi}{\lambda^2} + \frac{f_3(\lambda)}{\lambda^2},$$

$$s'(\pi, \lambda) = \cos \lambda \pi + Q \frac{\sin \lambda \pi}{\lambda} + \frac{f_4(\lambda)}{\lambda},$$

where $Q = \frac{1}{2} \int_0^\pi q(x) \, dx$, $f_1(\lambda), f_4(\lambda)$-are odd, $f_2(\lambda)$ and $f_3(\lambda)$ are even entire functions of exponential type not greater than $\pi$, square summable on the real axis. Taking into account these representations and using the Paley-Wiener theorem [8, p. 69], from (3) we obtain

$$\Delta(\lambda) = 1 - \gamma \lambda^2 \sin \pi \lambda + \lambda (\beta - \alpha \gamma + \gamma Q) \cos \pi \lambda +$$

$$(\beta Q - \alpha \beta - \alpha \gamma Q) \sin \pi \lambda + f(\lambda) + g(\lambda),$$

(4)

where $f(\lambda) = \int_0^\pi \tilde{f}(t) \cos \lambda t \, dt, g(\lambda) = \int_0^\pi \tilde{g}(t) \sin \lambda t \, dt, \tilde{f}(t), \tilde{g}(t) \in L_2[0, \pi]$. Using representation (4) and Rouche’s theorem, the following theorem can be proved by a standard method.

**Theorem 1.** For the eigenvalues $\mu_k (k = 0, \pm 1, \pm 2, \ldots)$ of the boundary value problem $P(\alpha, \beta, \gamma)$ at $|k| \to \infty$, the following asymptotic formula holds:

$$\mu_k = k + \frac{A}{\pi k} + \frac{\tau_k}{k},$$

(5)

where $\{\tau_k\} \in l_2$,

$$A = Q - \alpha + \frac{\beta}{\gamma}. $$

(6)

Along with the problem $P(\alpha, \beta, \gamma)$, we also consider the boundary value problem $P(\alpha, \bar{\beta}, \gamma)$ generated by the same equation (1) and the boundary conditions

$$y'(0) + \alpha y(0) = 0,$$

$$y(0) + \lambda \left[ \tilde{\beta} y(\pi) + \gamma y(\pi) \right] = 0.$$  

(7)

The spectrum of this problem will be denoted by $\{\tilde{\mu}_k\} (k = 0, \pm 1, \pm 2, \ldots)$. According to Theorem 1, this spectrum satisfies the asymptotic formula

$$\tilde{\mu}_k = k + \frac{\tilde{A}}{\pi k} + \frac{\tilde{\tau}_k}{k},$$  

(8)
at \(|k| \to \infty\), where \(\{\tilde{\tau}_k\} \in l_2\),

\[
\tilde{A} = Q - \alpha + \tilde{\beta} \gamma.
\]  
(9)

The sequences \(\{\mu_k\}, \{\tilde{\mu}_k\}\) and the number \(\gamma\) will be called the spectral data of a pair of boundary value problems \(P(\alpha, \beta, \gamma)\), \(P(\alpha, \tilde{\beta}, \gamma)\).

3. The Uniqueness Theorem

Consider two more problems with separated boundary conditions.

Problem \(P_1\):

\[
-y'' + q(x)y = \lambda^2 y (0 \leq x \leq \pi),
\]

\[
y'(0) + \alpha y(0) = 0,
\]

\[
y(\pi) = 0.
\]

Problem \(P_2\):

\[
-y'' + q(x)y = \lambda^2 y (0 \leq x \leq \pi),
\]

\[
y'(0) + \alpha y(0) = 0,
\]

\[
y'(\pi) = 0.
\]

The characteristic functions of these problems are

\[
\delta_1(\lambda) = c(\pi, \lambda) - \alpha s(\pi, \lambda),
\]  
(10)

\[
\delta_2(\lambda) = c'(\pi, \lambda) - \alpha s'(\pi, \lambda)
\]  
(11)

respectively.

Consider the following inverse problem.

**Inverse problem B.** Using the given spectral data of boundary value problems \(P(\alpha, \beta, \gamma)\) and \(P(\alpha, \tilde{\beta}, \gamma)\) construct the function \(q(x)\) in equation (1) and the coefficients \(\alpha, \beta, \tilde{\beta}\) in the boundary conditions (2) and (7).

The following uniqueness theorem is true.

**Theorem 2.** Boundary value problems \(P(\alpha, \beta, \gamma)\) and \(P(\alpha, \tilde{\beta}, \gamma)\) are uniquely determined by their spectral data.

**Proof.** Given the spectra \(\{\mu_k\}\) and \(\{\tilde{\mu}_k\}\) boundary value problems \(P(\alpha, \beta, \gamma)\) and \(P(\alpha, \tilde{\beta}, \gamma)\), we can uniquely determine the quantities \(A\) and \(\tilde{A}\), since, according to asymptotic formulas (5) and (8), we have

\[
A = \pi \lim_{k \to \infty} k (\mu_k - k), \quad \tilde{A} = \pi \lim_{k \to \infty} k (\tilde{\mu}_k - k).
\]

Then, by virtue of relations (6) and (9), the difference \(\beta - \tilde{\beta}\) is found as follows:

\[
\beta - \tilde{\beta} = \gamma \left( A - \tilde{A} \right).
\]
Using the spectral data of boundary value problems $P(\alpha, \beta, \gamma)$ and $\tilde{P}(\alpha, \tilde{\beta}, \gamma)$ construct the characteristic functions $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ in the form of an infinite product.

According to (3), (10) and (11)

$$\Delta(\lambda) = 1 + \lambda \left[ \beta \delta_1(\lambda) + \gamma \delta_2(\lambda) \right], \quad \tilde{\Delta}(\lambda) = 1 + \lambda \left[ \tilde{\beta} \delta_1(\lambda) + \gamma \delta_2(\lambda) \right].$$

Therefore, knowing the functions $\Delta(\lambda)$, $\tilde{\Delta}(\lambda)$ and the difference $\beta - \tilde{\beta}$, the characteristic function $\delta_1(\lambda)$ of the boundary value problem $P_1$ can be restored by the formula

$$\delta_1(\lambda) = \frac{\Delta(\lambda) - \tilde{\Delta}(\lambda)}{(\beta - \tilde{\beta}) \lambda}.$$

Using relation (4), for this function we obtain the following representation:

$$\delta_1(\lambda) = \cos \pi \lambda + \left( Q - \alpha \right) \frac{\sin \pi \lambda}{\lambda} + \frac{1}{\lambda} \int_0^\pi r(t) \sin \lambda t dt,$$

where $r(t) \in L_2[0, \pi]$. By virtue of Lemma 3.4.2 in [10], for the zeros $\lambda_n^{(1)} (n = 1, 2, \ldots)$ of the function $\delta_1(\lambda)$ at $n \to \infty$, the asymptotic formula

$$\lambda_n^{(1)} = n - \frac{1}{2} + \frac{Q - \alpha}{\pi n} + \left\{ \tau_n^{(1)} \right\} \in l_2.$$

From this formula we define the difference $Q - \alpha$ as follows:

$$Q - \alpha = \pi \lim_{n \to \infty} n \left( \lambda_n^{(1)} - n + \frac{1}{2} \right).$$

Knowing this difference, $A$, $\tilde{A}$ and $\gamma$, the quantities $\beta$ and $\tilde{\beta}$ are determined by the formulas

$$\beta = \gamma \left( A - Q + \alpha \right), \quad \tilde{\beta} = \gamma \left( \tilde{A} - Q + \alpha \right).$$

We reconstruct the characteristic function $\delta_2(\lambda)$ of the boundary value problem $P_2$ by the formula

$$\delta_2(\lambda) = \frac{\beta \tilde{\Delta}(\lambda) - \tilde{\beta} \Delta(\lambda) - \beta + \tilde{\beta}}{\gamma \left( \beta - \tilde{\beta} \right) \lambda}.$$

From the sequences of zeros of the functions $\delta_1(\lambda)$ and $\delta_2(\lambda)$ construct the potential $q(x)$ in (1) and the coefficient $\alpha$ in (2) by a well-known procedure (see, for example, [3], [9]).

The theorem is proved.

It is easy to see that the proof of the uniqueness theorem also contains an algorithm for solving the inverse problem B.
References