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UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR DIFFERENTIAL OPERATOR WITH SEMISEPARATED BOUNDARY CONDITIONS

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Abstract. *In the article we consider the Sturm-Liouville operator with semiseparated boundary conditions, one of which contains a spectral parameter. An asymptotic formula for the eigenvalues of the operator under consideration is given and a uniqueness theorem for the solution of the inverse problem of recovering the corresponding boundary value problems is proved.*

Keywords: Sturm-Liouville operator, eigenvalues, inverse problem, uniqueness theorem

Mathematics Subject Classification (2020): 34A55, 34B24, 34L05, 47E05

1. Introduction

Boundary-value problems with boundary conditions depending on the spectral parameter often arise in various fields of natural science and technology in the study of a number of problems, the construction of systems for the protection of devices against impact, vibrations of a string with a load at the end, torsional vibrations of a shaft with a flywheel at the end, vibrations of antennas loaded with concentrated capacities and inductances, etc. (see, for example, [1], [11] and the literature there). Inverse spectral problems associated with problems of this type also play an important role in the study of some nonlinear evolutionary equations of mathematical physics.

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Consider a boundary value problem generated on an interval $[0,\pi]$ by the Sturm-Liouville equation

$$-y''(x)+q(x)y(x)=\lambda^2 y(x) \tag{1}$$

and semiseparated boundary conditions of the form

$$\begin{gathered} y'(0)+\alpha y(0)=0, \\ y(0)+\lambda\left[\beta y(\pi)+\gamma y'(\pi)\right]=0, \end{gathered} \tag{2}$$

where $q(x)$ is a real function belonging to the space $L_2[0,\pi]$, λ is a spectral parameter, α, β, γ are real numbers. This problem will be denoted by $P(\alpha,\beta,\gamma)$. In this paper, we present an asymptotic formula for the eigenvalues of the considered Sturm-Liouville operator and prove a uniqueness theorem for the solution of the inverse problem of recovering the corresponding boundary value problems from spectral data. The spectra of two boundary value problems and a certain number are used as spectral data. Note that earlier the question of recovering differential operators for separated and nonseparated boundary conditions containing a spectral parameter was studied in [2], [4]-[7], [11]-[14] and other works.

2. Spectral Data of Boundary Value Problems

We denote by $c(x,\lambda)$, $s(x,\lambda)$, solutions of equation (1), satisfying the initial conditions

$$c(0,\lambda)=s'(0,\lambda)=1, c'(0,\lambda)=s(0,\lambda)=0.$$

For any x function $c(x,\lambda)$, $s(x,\lambda)$, $c'(x,\lambda)$, $s'(x,\lambda)$ are entire functions (of exponential type) of variable λ. The general solution of equation (1) is written in the form

$$y(x,\lambda)=A_1 c(x,\lambda)+A_2 s(x,\lambda),$$

where A_1, A_2- are arbitrary constants. Substituting this function into the boundary conditions (2) and using the last relations, we obtain for A_1 and A_2 the following system:

$$\begin{cases} A_1+\alpha A_2=0 \\ \left[\lambda\beta s(\pi,\lambda)+\lambda\gamma s'(\pi,\lambda)\right]A_1+\left[1+\lambda\beta c(\pi,\lambda)+\lambda\gamma c'(\pi,\lambda)\right]A_2=0. \end{cases}$$

For a number λ to be an eigenvalue of a boundary value problem $P(\alpha,\beta,\gamma)$, it is necessary and sufficient that the latter system has a nonzero solution. But this system has a nonzero solution if and only if its determinant is equal to zero. Therefore, the eigenvalues of the boundary value problem $P(\alpha,\beta,\gamma)$ coincide with the roots of the equation $\Delta(\lambda)=0$, where

$$\Delta(\lambda)=\begin{vmatrix} 1 & \alpha \\ \lambda\beta s(\pi,\lambda)+\lambda\gamma s'(\pi,\lambda) & 1+\lambda\beta c(\pi,\lambda)+\lambda\gamma c'(\pi,\lambda) \end{vmatrix}.$$

This function is called the characteristic function of the problem $P(\alpha,\beta,\gamma)$. Expanding the last determinant, we have

$$\Delta(\lambda)=1+\lambda\left[\beta(c(\pi,\lambda)-\alpha s(\pi,\lambda))+\gamma(c'(\pi,\lambda)-\alpha s'(\pi,\lambda))\right]. \tag{3}$$

Since $\Delta(0) = 1$, that $\lambda = 0$ is not an eigenvalue of the boundary value problem $P(\alpha, \beta, \gamma)$.

It is known [10, p. 47] that the functions $c(\pi, \lambda)$, $c'(\pi, \lambda)$, $s(\pi, \lambda)$ and $s'(\pi, \lambda)$ hold the following representations:

$$c(\pi, \lambda) = \cos\lambda\pi + Q\frac{\sin\lambda\pi}{\lambda} + \frac{f_1(\lambda)}{\lambda},$$

$$c'(\pi, \lambda) = -\lambda\sin\lambda\pi + Q\cos\lambda\pi + f_2(\lambda),$$

$$s(\pi, \lambda) = \frac{\sin\lambda\pi}{\lambda} - Q\frac{\cos\lambda\pi}{\lambda^2} + \frac{f_3(\lambda)}{\lambda^2},$$

$$s'(\pi, \lambda) = \cos\lambda\pi + Q\frac{\sin\lambda\pi}{\lambda} + \frac{f_4(\lambda)}{\lambda},$$

where $Q = \frac{1}{2}\int_0^\pi q(x)\,dx$, $f_1(\lambda)$, $f_4(\lambda)$-are odd, $f_2(\lambda)$ and $f_3(\lambda)$ are even entire functions of exponential type not greater than π, square summable on the real axis. Taking into account these representations and using the Paley-Wiener theorem [8, p. 69], from (3) we obtain

$$\Delta(\lambda) = 1 - \gamma\lambda^2\sin\pi\lambda + \lambda(\beta - \alpha\gamma + \gamma Q)\cos\pi\lambda +$$
$$+ (\beta Q - \alpha\beta - \alpha\gamma Q)\sin\pi\lambda + \lambda f(\lambda) + g(\lambda), \quad (4)$$

where $f(\lambda) = \int_0^\pi \tilde{f}(t)\cos\lambda t dt, g(\lambda) = \int_0^\pi \tilde{g}(t)\sin\lambda t dt$, $\tilde{f}(t), \tilde{g}(t) \in L_2[0, \pi]$. Using representation (4) and Rouche's theorem, the following theorem can be proved by a standard method.

Theorem 1. *For the eigenvalues $\mu_k (k = 0, \pm 0, \pm 1, \pm 2....)$ of the boundary value problem $P(\alpha, \beta, \gamma)$ at $|k| \to \infty$, the following asymptotic formula holds:*

$$\mu_k = k + \frac{A}{\pi k} + \frac{\tau_k}{k}, \quad (5)$$

where $\{\tau_k\} \in l_2$,

$$A = Q - \alpha + \frac{\beta}{\gamma}. \quad (6)$$

Along with the problem $P(\alpha, \beta, \gamma)$, we also consider the boundary value problem $P\left(\alpha, \tilde{\beta}, \gamma\right)$ generated by the same equation (1) and the boundary conditions

$$\begin{gathered} y'(0) + \alpha y(0) = 0, \\ y(0) + \lambda\left[\tilde{\beta} y(\pi) + \gamma y(\pi)\right] = 0. \end{gathered} \quad (7)$$

The spectrum of this problem will be denoted by $\{\tilde{\mu}_k\}$ $(k = 0, \pm 0, \pm 1, \pm 2....)$. According to Theorem 1, this spectrum satisfies the asymptotic formula

$$\tilde{\mu}_k = k + \frac{\tilde{A}}{\pi k} + \frac{\tilde{\tau}_k}{k}, \quad (8)$$

at $|k| \to \infty$, where $\{\tilde{\tau}_k\} \in l_2$,

$$\tilde{A} = Q - \alpha + \frac{\tilde{\beta}}{\gamma}. \tag{9}$$

The sequences $\{\mu_k\}$,$\{\tilde{\mu}_k\}$ and the number γ will be called the spectral data of a pair of boundary value problems $P(\alpha,\, \beta,\, \gamma)$, $P\left(\alpha,\, \tilde{\beta},\, \gamma\right)$.

3. The Uniqueness Theorem

Consider two more problems with separated boundary conditions.
Problem P_1:

$$-y'' + q(x)y = \lambda^2 y \left(0 \le x \le \pi\right),$$
$$y'(0) + \alpha y(0) = 0,$$
$$y(\pi) = 0.$$

Problem P_2:

$$-y'' + q(x)y = \lambda^2 y \left(0 \le x \le \pi\right),$$
$$y'(0) + \alpha y(0) = 0,$$
$$y'(\pi) = 0.$$

The characteristic functions of these problems are

$$\delta_1(\lambda) = c(\pi, \lambda) - \alpha s(\pi, \lambda), \tag{10}$$

$$\delta_2(\lambda) = c'(\pi, \lambda) - \alpha s'(\pi, \lambda) \tag{11}$$

respectively.

Consider the following inverse problem.

Inverse problem B. Using the given spectral data of boundary value problems $P(\alpha, \beta, \gamma)$ and $P\left(\alpha,\, \tilde{\beta}, \gamma\right)$ construct the function $q(x)$ in equation (1) and the coefficients $\alpha, \beta, \tilde{\beta}$ in the boundary conditions (2) and (7).

The following uniqueness theorem is true.

Theorem 2. *Boundary value problems $P(\alpha,\, \beta,\, \gamma)$ and $P\left(\alpha,\, \tilde{\beta},\, \gamma\right)$ are uniquely determined by their spectral data.*

Proof. Given the spectra $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$ boundary value problems $P(\alpha,\, \beta,\, \gamma)$ and $P(\alpha,\, \beta,\, \gamma)$, we can uniquely determine the quantities A and $\tilde{A}$, since, according to asymptotic formulas (5) and (8), we have

$$A = \pi \lim_{k\to\infty} k\left(\mu_k - k\right), \quad \tilde{A} = \pi \lim_{k\to\infty} k\left(\tilde{\mu}_k - k\right).$$

Then, by virtue of relations (6) and (9), the difference $\beta - \tilde{\beta}$ is found as follows:

$$\beta - \tilde{\beta} = \gamma\left(A - \tilde{A}\right).$$

Using the spectral data of boundary value problems $P(\alpha, \beta, \gamma)$ and $P\left(\alpha, \tilde{\beta}, \gamma\right)$ construct the characteristic functions $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ in the form of an infinite product. According to (3), (10) and (11)

$$\Delta(\lambda) = 1 + \lambda\left[\beta\delta_1(\lambda) + \gamma\delta_2(\lambda)\right], \quad \tilde{\Delta}(\lambda) = 1 + \lambda\left[\tilde{\beta}\delta_1(\lambda) + \gamma\delta_2(\lambda)\right].$$

Therefore, knowing the functions $\Delta(\lambda)$, $\tilde{\Delta}(\lambda)$ and the difference $\beta - \tilde{\beta}$, the characteristic function $\delta_1(\lambda)$ of the boundary value problem P_1 can be restored by the formula

$$\delta_1(\lambda) = \frac{\Delta(\lambda) - \tilde{\Delta}(\lambda)}{\left(\beta - \tilde{\beta}\right)\lambda}.$$

Using relation (4), for this function we obtain the following representation:

$$\delta_1(\lambda) = \cos\pi\lambda + (Q - \alpha)\frac{\sin\pi\lambda}{\lambda} + \frac{1}{\lambda}\int_0^{\pi} r(t)\sin\lambda t dt,$$

where $r(t) \in L_2[0, \pi]$. By virtue of Lemma 3.4.2 in [10], for the zeros $\lambda_n^{(1)}$ $(n = 1, 2, ...)$ of the function $\delta_1(\lambda)$ at $n \to \infty$, the asymptotic formula

$$\lambda_n^{(1)} = n - \frac{1}{2} + \frac{Q - \alpha}{\pi n} + \frac{\tau_n^{(1)}}{n}, \left\{\tau_n^{(1)}\right\} \in l_2.$$

From this formula we define the difference $Q - \alpha$ as follows:

$$Q - \alpha = \pi \lim_{n\to\infty} n\left(\lambda_n^{(1)} - n + \frac{1}{2}\right).$$

Knowing this difference, A, $\tilde{A}$ and γ, the quantities β and $\tilde{\beta}$ are determined by the formulas

$$\beta = \gamma(A - Q + \alpha), \tilde{\beta} = \gamma\left(\tilde{A} - Q + \alpha\right).$$

We reconstruct the characteristic function $\delta_2(\lambda)$ of the boundary value problem P_2 by the formula

$$\delta_2(\lambda) = \frac{\beta\tilde{\Delta}(\lambda) - \tilde{\beta}\Delta(\lambda) - \beta + \tilde{\beta}}{\gamma\left(\beta - \tilde{\beta}\right)\lambda}.$$

From the sequences of zeros of the functions $\delta_1(\lambda)$ and $\delta_2(\lambda)$ construct the potential $q(x)$in (1) and the coefficient α in (2) by a well-known procedure (see, for example, [3], [9]). The theorem is proved. ◄

It is easy to see that the proof of the uniqueness theorem also contains an algorithm for solving the inverse problem B.

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