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HARDY-LITTLEWOOD-STEIN-WEISS THEOREMS FOR RIESZ POTENTIALS IN MODIFIED MORREY SPACES

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Abstract. *The aim of this paper is to prove the Hardy-Littlewood-Stein-Weiss theorems for Riesz potentials in modified Morrey spaces* $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$.

Keywords: maximal operator, fractional maximal operator, Riesz potential, weighted modified Morrey space

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [14]), they are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ play an important role. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients and potential theory ([1]). An exposition of the Morrey spaces can be found in the book [11].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [3], [12], [16]. In [8] the boundedness of the maximal operator, singular integral operator and

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the potential operators in generalized variable exponent Morrey spaces were proved. In modified Morrey spaces the boundedness of the maximal operator, its commutator and Riesz potential operator were investigated by many authors (see, for example, [2], [9]).

Let $f \in L^1_{loc}(\mathbb{R}^n)$. As usual we define the Hardy-Littlewood maximal function of f, Mf, setting

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $B(x,t)$ denotes the open ball centered at x of radius t for $x \in \mathbb{R}^n$ and $t > 0$. $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$, such that $|B(x,t)| = \omega_n t^n$ and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

For $0 \le \alpha < n$, we define the fractional maximal function

$$M^\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|dy.$$

In the case $\alpha = 0$, we get $M^0 f = Mf$. The fractional maximal function is closely related to the Riesz potential operator

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

There are considerable number of results for Riesz potentials beyond the basic framework of the L_p-spaces, for example, the consideration of potentials with variable exponent [13], potentials mapping on L_p-spaces with variable exponent [6], potentials acting on Morrey spaces of variable exponent [5]. This list is by no means exhaustive, though gives an idea of some of the possible variations one can consider and obtain results analogous to those we have recorded here.

The results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [17], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [4].

Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces was proved in H.G. Hardy and J.E. Littlewood [10] in the one-dimensional case and to E.M. Stein in the case $n > 1$.

Throughout the paper we use the letter C for positive constants, independent of appropriate parameters and not necessarily the same at each occurrence. If $A \le CB$ and $B \le CA$, we write $A \sim B$ and say that A and B are equivalent.

We use the following notation. For $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

up to the equivalence of the norms

$$\|f\|_{L_p} \sim \sup_{\|g\|_{L_{p'}} \le 1} \left| \int_{\mathbb{R}^n} f(y) g(y) dy \right|$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r>0} r \left| \{x \in \mathbb{R}^n \ : \ |f(x)| > r\} \right|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Let $L_{p,\omega}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\omega}} = \|f \omega^{1/p}\|_{L_p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

and for $p = \infty$ the space $L_{\infty,\omega}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

Definition 1. *The weight function ω belongs to the class $A_p(\mathbb{R}^n)$ for $1 \le p < \infty$, if the following statement*

$$\sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega(y) dy \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $r > 0$

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \omega(y) dy \le C \operatorname*{ess\,sup}_{y \in B(x,r)} \frac{1}{\omega(y)}.$$

The following two theorems was proved in [18].

Theorem 1. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$. Then the operators M^α and I^α are bounded from $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,|\cdot|^\mu}(\mathbb{R}^n)$.*

Theorem 2. *Let $1 < p < \infty$ and $-n < \gamma < n(p-1)$, then the operator M is bounded on $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$.*

Definition 2. *Let $1 \le p < \infty$, $0 \le \lambda \le n$ and $[t]_1 = \min\{1, t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ Morrey space, and by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Definition 3. *Let $1 \le p < \infty$, $0 \le \lambda \le n$. We denote by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified weak Morrey space, as the space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ with finite norms*

$$\|f\|_{WL_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \; |\{y \in B(x,t) : \; |f(y)| > r\}| \right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \; |\{y \in B(x,t) : \; |f(y)| > r\}| \right)^{1/p}$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0}(\mathbb{R}^n) = W\widetilde{L}_{p,0}(\mathbb{R}^n),$$
$$L_{p,\lambda}(\mathbb{R}^n) \subset WL_{p,\lambda}(\mathbb{R}^n) \;\text{ and }\; \|f\|_{WL_{p,\lambda}} \le \|f\|_{L_{p,\lambda}},$$
$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset W\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \;\text{ and }\; \|f\|_{W\widetilde{L}_{p,\lambda}} \le \|f\|_{\widetilde{L}_{p,\lambda}}.$$

Now we define weighted modified Morrey spaces as follows.

Definition 4. *Let $1 \le p < \infty$, $0 \le \lambda \le n$, ω be a nonnegative measurable function on $\mathbb{R}^n$ and $[r]_1 = \min\{1, r\}$. We define weighted modified Morrey spaces $\widetilde{L}_{p,\lambda,\omega}(\mathbb{R}^n)$ as the set of all locally integrable functions f such that*

$$\|f\|_{\widetilde{L}_{p,\lambda,\omega}} = \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_{p,\omega}(B(x,r))} < \infty.$$

2. Maximal Operator in the Spaces $\widetilde{L}_{p,\lambda,\varphi}(\mathbb{R}^n)$

In this section we study the $\widetilde{L}_{p,\lambda,\varphi}$-boundedness of the maximal operator M.

Theorem 3. *[15] Let $1 < p < \infty$. Then the maximal operator M is bounded on $L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$.*

Theorem 4. *Let $1 < p < \infty$, $0 \le \lambda < n$ and $\varphi \in A_p(\mathbb{R}^n)$. Then M is bounded on $\widetilde{L}_{p,\lambda,\varphi}(\mathbb{R}^n)$.*

Proof. From the boundedness of maximal operator M on $L_{p,\varphi}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} (Mf(y))^p \varphi(y)dy \le C \int_{\mathbb{R}^n} |f(y)|^p \varphi(y)dy.$$

By the properties of A_p weights (see [7], Theorem 2.16, p. 407), we can easily see that for any $0 < \theta < 1$, $\psi = \varphi(M\chi_{B(x,r)})^\theta \in A_p(\mathbb{R}^n)$, then we get

$$\int_{B(x,t)} (Mf(y))^p \varphi(y)dy = \int_{\mathbb{R}^n} (Mf(y))^p \varphi(y)(M\chi_{B(x,r)})^\theta(y)dy \le$$

$$\le C_1 \int_{\mathbb{R}^n} |f(y)|^p \varphi(y)(M\chi_{B(x,r)})^\theta(y)dy.$$

As is known (see, [3], Lemma 2, p. 160), for all $t > 0$ and $x, y \in \mathbb{R}^n$

$$\left(\frac{t}{|x-y|+t}\right)^n \le M\chi_{B(x,t)}(y) \le \left(\frac{4t}{|x-y|+t}\right)^n.$$

Therefore we obtain the following inequalities

$$\int_{B(x,t)} (Mf(y))^p \varphi(y)dy \le$$

$$\le C\left(\int_{B(x,t)} |f(y)|^p\varphi(y)dy + \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}t)\setminus B(x,2^jt)} \frac{t^{n\theta}|f(y)|^p\varphi(y)dy}{(|x-y|+t)^{n\theta}}\right) \le$$

$$\le C\left([t]_1^\lambda \, \|f\|^p_{\widetilde{L}_{p,\lambda,\varphi}} + \|f\|^p_{\widetilde{L}_{p,\lambda,\varphi}} \sum_{j=0}^{\infty} \frac{[2^{j+1}t]_1^\lambda}{(2^j+1)^{n\theta}}\right) \le$$

$$\le C\,\|f\|^p_{\widetilde{L}_{p,\lambda,\varphi}} \left([t]_1^\lambda + \begin{cases} \left(2^\lambda\, t^\lambda \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-n\theta)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-n\theta j}\right)^{1/p}, & 0 < t < \frac{1}{2}, \\ \left(\sum\limits_{j=0}^{\infty} 2^{-n\theta j}\right)^{1/p}, & t \ge \frac{1}{2} \end{cases}\right) \le$$

$$\le C\,\|f\|^p_{\widetilde{L}_{p,\lambda,\varphi}} \left([t]_1^\lambda + \begin{cases} \left(C_1 t^\lambda + C_2 t^{n\theta}\right)^{1/p}, & 0 < t < \frac{1}{2}, \\ C_2^{1/p}, & t \ge \frac{1}{2} \end{cases}\right) \le C[t]_1^\lambda \, \|f\|^p_{\widetilde{L}_{p,\lambda,\varphi}},$$

which proves that M is bounded on $\widetilde{L}_{p,\lambda,\varphi}(\mathbb{R}^n)$. ◄

In the following theorem we give the necessary and sufficient condition for the boundedness of the maximal operator M on the spaces $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$.

Theorem 5. *Let $1 < p < \infty$ and $0 \le \lambda < n$. Then the maximal operator M is bounded on $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ if and only if $-n+\lambda \le \gamma < n(p-1)$.*

Proof. If part follows from Theorem 4, if $\varphi(x) = |x|^\gamma$. For only if part, let $0 \le \lambda < n$. The function $f(x) = \chi_{\{|x|<1\}}(x)$ belongs to $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ when $-n+\lambda \le \gamma$, but the maximal operator M of this function exists only when $\gamma < n(p-1)$. ◀

3. Riesz Potential Operator in the Spaces $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$

In this section we give the necessary and sufficient conditions for the boundedness of the Riesz potential operator from weighted modified Morrey spaces $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to weighted modified Morrey spaces $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.

First we give a lemma which we will use while proving our following theorem.

Lemma. *Let $0 \le \lambda < n$, $1 < p < \infty$, $-n+\lambda \le \gamma < n(p-1)$, $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ and define $f_t(x) =: f(tx)$, $t > 0$. Then the inequality*

$$\|f_t\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \le t^{-\frac{n+\gamma}{p}} [t]_1^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \tag{1}$$

holds.

Proof. Let $1 < p < \infty$, $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ and define $f_t(x) =: f(tx)$, $t > 0$. Then

$$\left([r]_1^{-\lambda} \int_{B(x,r)} |f_t(y)|^p |y|^\gamma dy\right)^{1/p} = t^{-\frac{n+\gamma}{p}} \left([r]_1^{-\lambda} \int_{B(x,tr)} |f(y)|^p |y|^\gamma dy\right)^{1/p} =$$

$$= t^{-\frac{n+\gamma}{p}} \left(\frac{[tr]_1}{[r]_1}\right)^{\frac{\lambda}{p}} \left([tr]_1^{-\lambda} \int_{B(x,tr)} |f(y)|^p |y|^\gamma dy\right)^{1/p} \le t^{-\frac{n+\gamma}{p}} [t]_1^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}.$$

Therefore we get

$$\|f_t\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \le t^{-\frac{n+\gamma}{p}} [t]_1^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}.$$

◀

Now we give the necessary and sufficinet conditions for the boundedness of Riesz potential operator I^α in the spaces $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$.

Theorem 6. *Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)$ and $\mu = \frac{q\gamma}{p}$. Then the operator I^α is bounded from $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ if and only if $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.*

Proof. *Sufficiency:* Let $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ and $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. Then

$$|I^\alpha f(x)| = \left(\int\limits_{B(x,t)} + \int\limits_{\mathbb{R}^n \setminus B(x,t)} \right) |f(y)||x-y|^{\alpha-n} dy \equiv J_1(x,t) + J_2(x,t).$$

First we estimate $J_1(x,t)$. By using Hölder's inequality we have

$$J_1(x,t) = \int\limits_{B(x,t)} |f(y)||x-y|^{\alpha-n} dy \le$$

$$\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \le C t^\alpha M f(x). \tag{2}$$

Now we estimate $J_2(x,t)$. By using Hölder's inequality we get

$$J_2(x,t) \le \int\limits_{\mathbb{R}^n \setminus B(x,t)} |f(y)||x-y|^{\alpha-n} dy \le$$

$$\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \le$$

$$\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \left\|\chi_{B(x,2^{j+1}t)}\right\|_{L_{p'(\cdot),|\cdot|^{\gamma/(1-p)}}} \left\|f\chi_{B(x,2^{j+1}t)}\right\|_{L_{p,|\cdot|^\gamma}} \le$$

$$\le C t^{\alpha-\frac{n}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \sum_{j=0}^{\infty} 2^{j\left(\alpha-\frac{n}{p}\right)} [2^j t]_1^{\frac{\lambda}{p}} \le C t^{\alpha-\frac{n}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \times$$

$$\times \begin{cases} \left(2^\lambda t^\lambda \sum\limits_{j=0}^{\log_2[\frac{1}{2t}]} 2^{j\left(\alpha-\frac{n-\lambda}{p}\right)} + \sum\limits_{j=\log_2[\frac{1}{2t}]+1}^{\infty} 2^{j\left(\alpha-\frac{n}{p}\right)}\right)^{1/p}, & 0<t<1, \\ \left(\sum\limits_{j=0}^{\infty} 2^{j\left(\alpha-\frac{n}{p}\right)}\right)^{1/p}, & t \ge 1 \end{cases} \le$$

$$\le C t^{\alpha-\frac{n}{p}} [t]_1^{\frac{\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}.$$

Thus

$$J_2(x,t) \le C t^{\alpha-\frac{n}{p}} [t]_1^{\frac{\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}. \tag{3}$$

So, from (2) and (3) we have

$$|I^\alpha f(x)| \le C t^\alpha M f(x) + C t^{\alpha-\frac{n}{p}} [t]_1^{\frac{\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \le$$

$$\le C \min\left\{ t^\alpha M f(x) + C t^{\alpha-\frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}},\ t^\alpha M f(x) + C t^{\alpha-\frac{n}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \right\}.$$

Minimizing with respect to $t = \left[(Mf(x))^{-1}\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}\right]^{\frac{p}{n-\lambda}}|x|^{-\frac{\gamma}{n-\lambda}}$ or $t = \left[(Mf(x))^{-1}\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}\right]^{\frac{p}{n}}|x|^{-\frac{\gamma}{n}}$ we obtain

$$|I^\alpha f(x)| \le C\min\left\{\left(\frac{Mf(x)}{\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}}\right)^{1-\frac{p\alpha}{n-\lambda}}|x|^{-\frac{\gamma\alpha}{n-\lambda}}, \left(\frac{Mf(x)}{\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}}\right)^{1-\frac{p\alpha}{n}}|x|^{-\frac{\gamma\alpha}{n}}\right\}.$$

Hence, from Theorem 5, we get

$$\int\limits_{B(x,t)}|I^\alpha f(y)|^q|y|^\mu dy \le C\,\|f\|^{q-p}_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}\int\limits_{B(x,t)}(Mf(y))^p\,|y|^\gamma dy \le$$

$$\le C[t]_1^\lambda\,\|f\|^{q-p}_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}\,\|f\|^{p}_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} = C[t]_1^\lambda\,\|f\|^{q}_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}.$$

Therefore $I^\alpha f \in \widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ and

$$\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}.$$

Necessity: Let $1 < p < \frac{n-\lambda}{\alpha}$ and I^α be bounded from $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Define $f_t(x) =: f(tx)$, $t > 0$. Using inequality (1), since

$$I^\alpha f_t(x) = t^{-\alpha}I^\alpha f(tx),$$

we obtain

$$\left([r]_1^{-\lambda}\int_{B(x,r)}|I^\alpha f_t(y)|^q\,|y|^\mu dy\right)^{1/q} = t^{-\alpha}\left([r]_1^{-\lambda}\int_{B(x,r)}|I^\alpha f(ty)|^q\,|y|^\mu dy\right)^{1/q} =$$

$$= t^{-\alpha-\frac{n+\mu}{q}}\left(\frac{[tr]_1}{[r]_1}\right)^{\frac{\lambda}{q}}\left([tr]_1^{-\lambda}\int_{B(x,tr)}|I^\alpha f(y)|^q\,|y|^\mu dy\right)^{1/q} \le t^{-\alpha-\frac{n+\mu}{q}}[t]_1^{\frac{\lambda}{q}}\,\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}}.$$

Therefore we get

$$\|I^\alpha f_t\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le t^{-\alpha-\frac{n+\mu}{q}}[t]_1^{\frac{\lambda}{q}}\,\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}}.$$

Since the operator I^α is bounded from $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, we have

$$\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le t^{\alpha+\frac{n+\mu}{q}}[t]_1^{-\frac{\lambda}{q}}\,\|I^\alpha f_t\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le$$

$$\le Ct^{\alpha+\frac{n+\mu}{q}}[t]_1^{-\frac{\lambda}{q}}\|f_t\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}} \le Ct^{\alpha+\frac{n+\mu}{q}-\frac{n+\gamma}{p}}[t]_1^{\frac{\lambda}{p}-\frac{\lambda}{q}}\|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}, \tag{4}$$

where C depends on p, q, λ, γ, μ and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\lambda}$, from the inequality (4), $\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t \to 0$.

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n}$, from the inequality (4), $\|I^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t \to \infty$. Therefore $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. ◀

Corollary. *Let* $0 < \alpha < n$, $0 \le \lambda < n-\alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$. *Then the fractional maximal operator* M^α *is bounded from* $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ *if and only if* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.

Proof. The sufficiency part follows from the inequality $M^\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} I^\alpha |f|(x)$. For necessity part, let $1 < p < \frac{n-\lambda}{\alpha}$ and M^α be bounded from $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Define $f_t(x) =: f(tx)$, $t > 0$.

Since

$$M^\alpha f_t(x) = t^{-\alpha} M^\alpha f(tx),$$

and by (1), then

$$\left([r]_1^{-\lambda} \int_{B(x,r)} |M^\alpha f_t(y)|^q \, |y|^\mu dy\right)^{1/q} = t^{-\alpha} \left([r]_1^{-\lambda} \int_{B(x,r)} |M^\alpha f(ty)|^q \, |y|^\mu dy\right)^{1/q} =$$

$$= t^{-\alpha - \frac{n+\mu}{q}} \left(\frac{[tr]_1}{[r]_1}\right)^{\frac{\lambda}{q}} \left([tr]_1^{-\lambda} \int_{B(x,tr)} |M^\alpha f(y)|^q \, |y|^\mu dy\right)^{1/q} \le$$

$$\le t^{-\alpha - \frac{n+\mu}{q}} [t]_1^{\frac{\lambda}{q}} \, \|M^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}}.$$

Therefore we get

$$\|M^\alpha f_t\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le t^{-\alpha - \frac{n+\mu}{q}} [t]_1^{\frac{\lambda}{q}} \, \|M^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}}.$$

Since the operator M^α is bounded from $\widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, we have

$$\|M^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} \le C t^{\alpha + \frac{n+\mu}{q} - \frac{n+\gamma}{p}} [t]_1^{\frac{\lambda}{p} - \frac{\lambda}{q}} \|f\|_{\widetilde{L}_{p,\lambda,|\cdot|^\gamma}}, \tag{5}$$

where C depends on p,q,λ,γ,μ and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\lambda}$, from the inequality (5), $\|M^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t \to 0$.

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n}$, from the inequality (5), $\|M^\alpha f\|_{\widetilde{L}_{q,\lambda,|\cdot|^\mu}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t \to \infty$. Therefore $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. ◀

References

1. Adams D.R. A note on Riesz potentials. *Duke Math.*, 1975, **42**, pp. 765–778.
2. Aykol C., Armutcu H., Omarova M.N. Maximal commutator and commutator of maximal function on modified Morrey spaces. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics,* 2016, **36** (1), pp. 29–35.
3. Burenkov V.I., Guliyev H.V. Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. *Studia Math.*, 2004, **163** (2), pp. 157–176.

4. Chiarenza F., Frasca M. Morrey spaces and Hardy–Littlewood maximal function. *Rend. Math.*, 1987, **7**, pp. 273–279.
5. Duong X.T., Yan L.X. On commutators of fractional integrals. *Proc. Amer. Math. Soc.*, 2004, **132** (12), pp. 3549–3557.
6. Futamura T., Mizuta Y. Continuity properties of Riesz potentials for functions in $l_{p(\cdot)}$ of variable exponent. *Math. Inequal. Appl.*, 2005, **8**, pp. 619-631.
7. Garcia-Cuerva J., Rubio de Francia J.L. *Weighted Norm Inequalities and Related Topics.* North-Holland, Amsterdam, New York, Oxford, 1985.
8. Guliev V.S., Hasanov J.J., Samko S.G. Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. *Math. Scand.*, 2010, **107**, pp. 285-304.
9. Guliyev V.S., Hasanov J.J., Zeren Y. Necessary and sufficient conditions for the boundedness of B-Riesz potential in modifed Morrey spaces. *J. Math. Inequal*, 2011, **5** (4), pp. 491-506.
10. Hardy H.G., Littlewood J.E. Some properties of fractional integrals. *I. Math. Z.*, 1928, **27** (4), pp. 565–606.
11. Kufner A., John O., Fuçik S. *Function Spaces.* Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences, Prague, 1977.
12. Mizuhara T. *Boundedness of some classical operators on generalized Morrey spaces.* Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 1991.
13. Mizuta Y., Shimomura T. Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity. *J. Math. Anal. Appl.*, 2005, **311**, pp. 268-288.
14. Morrey C.B. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 1938, **43**, pp. 126-166.
15. Muckenhoupt B. Weighted norm inequalities for the Hardy-Littlewood maximal function. *Trans. Amer. Math. Soc.*, 1972, **165**, pp. 207-226.
16. Nakai E. Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. *Math. Nachr.*, 1994, **166**, pp. 95-103.
17. Peetre J. On the theory of $\mathcal{L}^{p,\lambda}$ spaces. *J. Funct. Anal.*, 1969, **4**, pp. 71-87.
18. Stein E.M. *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals.* Princeton Univ. Press, Princeton, NJ, 1993.