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QUADRATURE FORMULAS FOR SOME CLASSES OF CURVILINEAR INTEGRALS

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Abstract. *Quadrature formulas for simple-layer and double-layer potentials are established, and error estimates for these quadrature formulas are obtained.*

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1. Introduction

As is known, the problems of seeking for a solution of boundary value problems for the Helmholtz equation $\Delta u+k^2u=0$ in two-dimensional space in the form of combination of simple-layer and double-layer potentials are reduced to the curvilinear integral equation (see [2]) depending on the operators

$$(S\rho)\left(x\right)=2\int_L \Phi\left(x,\,y\right)\,\rho\left(y\right)dL_y,\quad x=\left(x_1,\,x_2\right)\in L, \tag{1}$$

and

$$(K\rho)\left(x\right)=2\int_L \frac{\partial\Phi\left(x,\,y\right)}{\partial\nu\left(y\right)}\,\rho\left(y\right)dL_y,\quad x=\left(x_1,\,x_2\right)\in L, \tag{2}$$

where Δ is a Laplace operator, k is a wave number with $Imk\geq 0$, $L\subset\mathbb{R}^2$ is a simple closed Lyapunov curve, $\rho(y)$ is a continuous function on the curve L, $\nu(y)$ is an outer unit normal at the point $y\in L$, $\Phi(x,\,y)$ is a fundamental solution of the Helmholtz equation, i.e.

$$\Phi(x,\,y)=\begin{cases}\frac{1}{2\pi}\ln\frac{1}{|x-y|} & for\;\; k=0,\\ \frac{i}{4}H_0^{(1)}\left(k\left|x-y\right|\right) & for\;\; k\neq 0,\end{cases}$$

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where $H_0^{(1)}$ is a zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}(z) = J_0(z) + i\, N_0(z)$,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is a Bessel function of zero degree,

$$N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C\right) J_0(z) + \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is a Neumann function of zero degree (see [3]), and C is an Euler's constant.

Note that it is impossible in many cases to find an exact solution of the above integral equations. Therefore, there is an interest in the study of approximate solutions of these integral equations by the collocation method, and for this, you first need to construct quadratic formulas for simple-layer and double-layer potentials. It should be noted that in [9], the quadratic formulas for simple-layer and double-layer logarithmic potentials (i.e. for $k = 0$) have been obtained, and in [4], [5], cubature formulas for simple-layer and double-layer acoustic potentials have been constructed. Besides, based on these quadrature and cubature formulas, the justification of collocation method for integral equations of boundary value problems for the Helmholtz equation has been given in [1], [6], [8], [10]. This work deals with the construction of quadrature formula for the integrals (1) and (2) when $k \neq 0$.

2. Quadrature Formula for the Integral (1)

Let the curve L be defined by the parametric equation $x(t) = (x_1(t), x_2(t))$, $t \in [a, b]$. Let's divide the interval $[a, b]$ into $n > 2M_0(b-a)/d$ equal parts: $t_p = a + \frac{(b-a)\,p}{n}$, $p = \overline{0, n}$, where $M_0 = \max_{t\in[a,b]} \sqrt{(x_1'(t))^2 + (x_2'(t))^2} < +\infty$ (see [11, p. 560]) and d is a standart radius (see [12, p. 400]). As control points, we consider $x(\tau_p)$, $p = \overline{1, n}$, where $\tau_p = a + \frac{(b-a)(2p-1)}{2n}$. Then the curve L is divided into elementary parts: $L = \bigcup_{p=1}^{n} L_p$, where $L_p = \{\, x(t) : \ t_{p-1} \le t \le t_p\}$.

As is known (see [9]),

(1) $\forall p \in \{1, 2, ..., n\}$: $r_p(n) \sim R_p(n)$, where $r_p(n) = \min\{\, |x(\tau_p) - x(t_{p-1})|\,, |x(t_p) - x(\tau_p)|\,\}$, $R_p(n) = \max\{\, |x(\tau_p) - x(t_{p-1})|\,, |x(t_p) - x(\tau_p)|\,\}$, and the expression $a(n) \sim b(n)$ means $C_1 \le \frac{a(n)}{b(n)} \le C_2$, where C_1 and C_2 are positive constants independent of n.

(2) $\forall p \in \{1, 2, ..., n\} : R_p(n) \le d/2$;

(3) $\forall p, j \in \{1, 2, ..., n\} : r_j(n) \sim r_p(n)$;

(4) $r(n) \sim R(n) \sim \frac{1}{n}$, where $R(n) = \max_{p=\overline{1,n}} R_p(n)$, $r(n) = \min_{p=\overline{1,n}} r_p(n)$.

In the sequel, we will call this kind of division a division of the curve L into "regular" elementary parts.

Proceeding as in the proof of Lemma 2.1 in [7], we can prove the validity of the following lemma.

Lemma 1. *There exist the constants* $C_0' > 0$ *and* $C_1' > 0$, *independent of* n, *such that for* $\forall p, j \in \{1, 2, ..., n\}$, $j \neq p$, *and* $\forall y \in L_j$ *the following inequalities hold:*

$$C_0' \; |y - x(\tau_p)| \leq |x(\tau_j) - x(\tau_p)| \leq C_1' \; |y - x(\tau_p)| \, .$$

Denote by $C(L)$ a space of all continuous functions on L with the norm $\|\rho\|_\infty = \max_{x \in L} |\rho(x)|$, and introduce for the function $\varphi(x) \in C(L)$ a modulus of continuity of the form

$$\omega(\varphi, \delta) = \delta \sup_{\tau \geq \delta} \frac{\bar{\omega}(\varphi, \tau)}{\tau}, \quad \delta > 0,$$

where $\bar{\omega}(\varphi, \tau) = \max\limits_{\substack{|x-y| \leq \tau \\ x, y \in L}} |\varphi(x) - \varphi(y)| \, .$

Let

$$\Phi_n(x, y) = \frac{i}{4} H_{0,n}^{(1)}(k|x-y|), \quad x, y \in L, \quad x \neq y,$$

where

$$H_{0,n}^{(1)}(z) = J_{0,n}(z) + i \, N_{0,n}(z),$$

$$J_{0,n}(z) = \sum_{m=0}^{n} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

and

$$N_{0,n}(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C\right) J_{0,n}(z) + \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m}.$$

Theorem 1. *Let* $L \subset \mathbb{R}^2$ *be a simple closed Lyapunov curve of order* $0 < \alpha \leq 1$ *and* $\rho \in C(L)$. *Then the expression*

$$(S_n\rho)(x(\tau_p)) = \frac{2(b-a)}{n} \sum_{\substack{j=1 \\ j \neq p}}^{n} \Phi_n(x(\tau_p), x(\tau_j)) \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2} \, \rho(x(\tau_j))$$

is a quadrature formula at the control points $x(\tau_p)$, $p = \overline{1, n}$, *for the integral (1), and the following estimates hold:*

$$\max_{p=\overline{1,n}} |(S\rho)(x(\tau_p)) - (S_n\rho)(x(\tau_p))| \leq M \left(\omega(\rho, 1/n) + \|\rho\|_\infty \frac{1}{n^\alpha}\right) \; for \; 0 < \alpha < 1,$$

$$\max_{p=\overline{1,n}} |(S\rho)(x(\tau_p)) - (S_n\rho)(x(\tau_p))| \leq M \left(\omega(\rho, 1/n) + \|\rho\|_\infty \frac{\ln n}{n}\right) \; for \; \alpha = 1.$$

Hereinafter M *denotes a positive constant which can be different in different inequalities.*

Proof. It is not difficult to see that

$$(S\rho)(x(\tau_p)) - (S_n\rho)(x(\tau_p)) = 2\int_{L_p} \Phi(x(\tau_p), y)\, \rho(y)\, dL_y +$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} (\Phi(x(\tau_p), y) - \Phi_n(x(\tau_p), x(\tau_j)))\, \rho(y)\, dL_y +$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} \Phi_n(x(\tau_p), x(\tau_j))\, (\rho(y) - \rho(x(\tau_j)))\, dL_y +$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{t_{j-1}}^{t_j} \Phi_n(x(\tau_p), x(\tau_j)) \times$$

$$\times\left(\sqrt{(x_1'(t))^2 + (x_2'(t))^2} - \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}\right) \rho(x(\tau_j))\, dt.$$

Let's denote the terms in the last equality by $h_1^n(x(\tau_p))$, $h_2^n(x(\tau_p))$, $h_3^n(x(\tau_p))$ and $h_4^n(x(\tau_p))$, respectively.

It is evident that

$$|J_0(k|x-y|)| = \left|\sum_{m=0}^{\infty} \frac{(-1)^m}{(m\,!)^2}\left(\frac{k|x-y|}{2}\right)^{2m}\right| \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{(|k|\; diamL)^{2m}}{4^m (m\,!)^2} = M_1, \quad \forall x, y \in L, \tag{3}$$

and

$$\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\frac{k|x-y|}{2}\right)^{2m}\right| \leq$$

$$\leq \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(|k|\; diamL)^{2m}}{4^m (m!)^2} = M_2, \quad \forall x, y \in L. \tag{4}$$

Consequently,

$$|\Phi(x, y)| \leq M\;|\ln|x-y||, \quad \forall x, y \in L, \quad x \neq y. \tag{5}$$

Then, using the formula for the curvilinear integral, we obtain

$$|h_1^n(x(\tau_p))| \leq 2\;\|\rho\|_\infty \int_{L_p} |\Phi(x(\tau_p), y)|\, dL_y \leq$$

$$\leq M\;\|\rho\|_\infty \int_0^{R(n)} |\ln\tau|\; d\tau \leq M\;\|\rho\|_\infty\, R(n)\; |\ln R(n)|\,.$$

Let $y \in L_j$ and $j \neq p$. Taking into account Lemma 1, we have

$$||x(\tau_p) - y|^q - |x(\tau_p) - x(\tau_j)|^q| = ||x(\tau_p) - y| - |x(\tau_p) - x(\tau_j)|| \times$$
$$\times \left(|x(\tau_p) - y|^{q-1} + |x(\tau_p) - y|^{q-2} |x(\tau_p) - x(\tau_j)| + ... + |x(\tau_p) - x(\tau_j)|^{q-1}\right) \leq$$
$$\leq M\, q\, |x(\tau_j) - y|\, |x(\tau_p) - y|^{q-1} \leq M\, q\, R(n)\, (diamL)^{q-1} \tag{6}$$

and

$$|\ln(k|x(\tau_p) - y|) - \ln(k|x(\tau_p) - x(\tau_j)|)| = \left|\ln \frac{|x(\tau_p) - x(\tau_j)|}{|x(\tau_p) - y|}\right| =$$
$$= \left|\ln\left(1 + \frac{|x(\tau_p) - x(\tau_j)| - |x(\tau_p) - y|}{|x(\tau_p) - y|}\right)\right| \leq$$
$$\leq \left|\ln\left(1 + \frac{|x(\tau_j) - y|}{|x(\tau_p) - y|}\right)\right| \leq M \frac{R(n)}{|x(\tau_p) - y|}, \tag{7}$$

where $q \in \mathbb{N}$. Then, by the inequalities (3), (4), (6) and (7), we obtain

$$|\Phi(x(\tau_p), y) - \Phi(x(\tau_p), x(\tau_j))| \leq$$
$$\leq \frac{1}{4}\left|\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\left(\frac{k|x(\tau_p) - y|}{2}\right)^{2m} - \left(\frac{k|x(\tau_p) - x(\tau_j)|}{2}\right)^{2m}\right)\right| +$$
$$+\frac{1}{2\pi}\left|\left(\ln\frac{k|x(\tau_p) - x(\tau_j)|}{2} + C\right) \times\right.$$
$$\left.\times \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\left(\frac{k|x(\tau_p) - y|}{2}\right)^{2m} - \left(\frac{k|x(\tau_p) - x(\tau_j)|}{2}\right)^{2m}\right)\right| +$$
$$+\frac{1}{2\pi}\left|(\ln(k|x(\tau_p) - y|) - \ln(k|x(\tau_p) - x(\tau_j)|)) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{k|x(\tau_p) - y|}{2}\right)^{2m}\right| +$$
$$+\frac{1}{4}\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\left(\frac{k|x(\tau_p) - y|}{2}\right)^{2m} - \left(\frac{k|x(\tau_p) - x(\tau_j)|}{2}\right)^{2m}\right)\right| \leq$$
$$\leq MR(n)\sum_{m=0}^{\infty}\frac{|k|^{2m}(diamL)^{2m-1}}{4^m m!\,(m-1)!} +$$
$$+MR(n)\,|\ln(k|x(\tau_p) - x(\tau_j)|)|\sum_{m=0}^{\infty}\frac{|k|^{2m}(diamL)^{2m-1}}{4^m m!\,(m-1)!} +$$
$$+\frac{MR(n)}{|x(\tau_p) - y|}\sum_{m=0}^{\infty}\frac{(|k|\, diamL)^{2m}}{4^m (m!)^2} +$$
$$+MR(n)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{|k|^{2m}(diamL)^{2m-1}}{4^m m!\,(m-1)!} \leq \frac{MR(n)}{|x(\tau_p) - y|}.$$

Besides, in view of the inequalities

$$\left|J_0\left(k\left|x-y\right|\right)-J_{0,n}\left(k\left|x-y\right|\right)\right|\leq\sum_{m=n+1}^{\infty}\frac{\left|k\right|^{2m}\left|x-y\right|^{2m}}{4^m\left(m!\right)^2}\leq\frac{M}{(n+1)\ !},\quad\forall x,y\in L,\quad(8)$$

and

$$\left|N_0\left(k\left|x-y\right|\right)-N_{0,n}\left(k\left|x-y\right|\right)\right|=$$
$$=\frac{2}{\pi}\left|\ln\frac{k\left|x-y\right|}{2}+C\right|\left|J_0\left(k\left|x-y\right|\right)-J_{0,n}\left(k\left|x-y\right|\right)\right|+$$
$$+\sum_{m=n+1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left|k\right|^{2m}\left|x-y\right|^{2m}}{4^m\left(m!\right)^2}\leq\frac{M\left|\ln\left|x-y\right|\right|}{(n+1)\ !},\quad\forall x,y\in L,\qquad(9)$$

we have

$$\left|\Phi\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)-\Phi_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\right|\leq$$
$$\leq\frac{M\left|\ln\left|x\left(\tau_p\right)-\,x\left(\tau_j\right)\right|\right|}{(n+1)\ !}\leq\frac{M}{(n+1)\ !\ \left|x\left(\tau_p\right)-\,y\right|}.$$

Then we get

$$\left|\Phi\left(x\left(\tau_p\right),\,y\right)-\Phi_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\right|\leq\left|\Phi\left(x\left(\tau_p\right),\,y\right)-\Phi\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\right|+$$
$$+\left|\Phi\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)-\Phi_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\right|\leq\frac{M}{\left|x\left(\tau_p\right)-y\right|}\left(R\left(n\right)+\frac{1}{(n+1)\ !}\right).$$

Consequently,

$$\left|h_2^n\left(x\left(\tau_p\right)\right)\right|\leq M\ \left\|\rho\right\|_\infty\left(R\left(n\right)+\frac{1}{(n+1)\ !}\right)\int_{r(n)}^{diam\,L}\frac{d\tau}{\tau}\leq$$
$$\leq M\ \left\|\rho\right\|_\infty\left(R\left(n\right)+\frac{1}{(n+1)\ !}\right)\left|\ln R\left(n\right)\right|.$$

From the inequality (5) it follows that the integral

$$\int_L\left|\Phi\left(x,\,y\right)\right|\,dL_y$$

converges as an improper integral and

$$\int_L\left|\Phi\left(x,\,y\right)\right|\,dL_y\leq M,\quad\forall x\in L.$$

Then from the inequalities (8) and (9) we obtain

$$\int_L\left|\Phi_n\left(x,\,y\right)\right|\,dL_y\leq\int_L\left|\Phi\left(x,\,y\right)\right|\,dL_y+\int_L\left|\Phi\left(x,\,y\right)-\Phi_n\left(x,\,y\right)\right|\,dL_y\leq$$
$$\leq M+\frac{M}{(n+1)\ !}\int_L\left|\ln\left|x-y\right|\right|\,dL_y\leq M,\quad\forall x\in L,\quad\forall n\in \mathrm{N}.$$

As a result, taking into account Lemma 1, we obtain

$$|h_3^n(x(\tau_p))| \le M\,\omega(\rho, R(n)) \int_L |\Phi_n(x(\tau_p), y)|\, dL_y \le M\omega\ (\rho, R(n)).$$

Obviously,

$$\left|\sqrt{(x_1'(t))^2+(x_2'(t))^2}-\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\right| \le M\,(R(n))^\alpha, \quad \forall t\in[t_{j-1},t_j]. \quad (10)$$

Let $y \in L_j$ and $j \neq p$. By Lemma 1 and the inequalities (3) and (4), we have

$$|J_{0,n}(k\,|x(\tau_p)-x(\tau_j)|)| \le \sum_{m=0}^{n} \frac{(|k|\ diamL)^{2m}}{4^m\,(m\,!)^2} \le M_1, \quad \forall n \in \mathrm{N},$$

and

$$|N_{0,n}(k\,|x(\tau_p)-x(\tau_j)|)| \le$$

$$\le \frac{2M_1}{\pi}\left|\ln\frac{k\,(x(\tau_p)-x(\tau_j))}{2}+C\right| + \sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(|k|\,diamL)^{2m}}{4^m\,(m!)^2} \le$$

$$\le \frac{2M_1}{\pi}\left|\ln\frac{k\,(x(\tau_p)-x(\tau_j))}{2}+C\right| + M_2 \le M\,|\ln|x(\tau_p)-y||, \quad \forall n \in \mathrm{N}.$$

Consequently,

$$|\Phi_n(x(\tau_p), x(\tau_j))| \le M\ |\ln|x(\tau_p)-y||, \quad \forall n \in \mathrm{N}.$$

Hence it follows that

$$|h_4^n(x(\tau_p))| \le M\,\|\rho\|_\infty\,(R(n))^\alpha \sum_{\substack{j=1\\ j\neq p}}^{n} \int_{t_{j-1}}^{t_j} |\Phi_n(x(\tau_p), x(\tau_j))|\ dt \le$$

$$\le M\,\|\rho\|_\infty\,(R(n))^\alpha \sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} |\Phi_n(x(\tau_p), x(\tau_j))|\, dL_y \le$$

$$\le M\,\|\rho\|_\infty\,(R(n))^\alpha \sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} |\ln|x(\tau_p)-y||\ dL_y \le$$

$$\le M\,\|\rho\|_\infty\,(R(n))^\alpha \int_L |\ln|x(\tau_p)-y||\ dL_y \le M\,\|\rho\|_\infty\,(R(n))^\alpha.$$

Finally, summing up the estimates obtained for the expressions $h_1^n(x(\tau_p))$, $h_2^n(x(\tau_p))$, $h_3^n(x(\tau_p))$ and $h_4^n(x(\tau_p))$ and taking into account the relation $R(n) \sim \frac{1}{n}$, we get the validity of Theorem 1. ◄

3. Quadrature Formula for the Integral (2)

Now let's construct the quadrature formula for the integral (2). It is not difficult to show that

$$\frac{\partial \Phi_n(x,y)}{\partial \nu(y)} = \frac{i}{4}\left(\frac{\partial J_{0,n}(k|x-y|)}{\partial \nu(y)} + i\,\frac{\partial N_{0,n}(k|x-y|)}{\partial \nu(y)}\right),$$

where

$$\frac{\partial J_{0,n}(k|x-y|)}{\partial \nu(y)} = (y-x,\,\nu(y))\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}$$

and

$$\frac{\partial N_{0,n}(k|x-y|)}{\partial \nu(y)} = \frac{2}{\pi}\left(\ln\frac{k\,|x-y|}{2}+C\right)\frac{\partial J_{0,n}(k|x-y|)}{\partial \nu(y)} +$$

$$+\frac{2\,(y-x,\,\nu(y))}{\pi\,|x-y|^2}J_{0,n}(k|x-y|)+$$

$$+(y-x,\,\nu(y))\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}.$$

Let's divide the curve L into regular elementary parts: $L=\bigcup_{p=1}^{n}L_p$. Then the following theorem is true.

Theorem 2. *Let $L\subset\mathbb{R}^2$ be a simple closed Lyapunov curve of order $0<\alpha\le 1$ and $\rho\in C(L)$. Then the expression*

$$(K_n\rho)(x(\tau_p)) = \frac{2(b-a)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n}\frac{\partial \Phi_n(x(\tau_p),\,x(\tau_j))}{\partial \nu(x(\tau_j))}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,\rho(x(\tau_j))$$

is a quadrature formula at the control points $x(\tau_p)$, $p=\overline{1,n}$, for the integral (2), and the following estimate holds:

$$\max_{p=\overline{1,n}}|(K\rho)(x(\tau_p))-(K_n\rho)(x(\tau_p))| \le M\left(\omega(\rho,\,1/n)+\|\rho\|_\infty\frac{\ln n}{n^\alpha}\right).$$

Proof. It is not difficult to see that

$$(K\rho)(x(\tau_p))-(K_n\rho)(x(\tau_p)) = 2\int_{L_p}\frac{\partial \Phi(x(\tau_p),\,y)}{\partial \nu(y)}\rho(y)\,dL_y+$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{L_j}\left(\frac{\partial \Phi(x(\tau_p),\,y)}{\partial \nu(y)}-\frac{\partial \Phi_n(x(\tau_p),\,x(\tau_j))}{\partial \nu(x(\tau_j))}\right)\rho(y)\,dL_y+$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{L_j}\frac{\partial\Phi_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\,\left(\rho\left(y\right)-\rho\left(x\left(\tau_j\right)\right)\right)dL_y+$$

$$+2\sum_{\substack{j=1\\ j\neq p}}^{n}\int_{t_{j-1}}^{t_j}\frac{\partial\Phi_n\left(x(\tau_p),\,x(\tau_j)\right)}{\partial\nu\left(x(\tau_j)\right)}\times$$

$$\times\left(\sqrt{(x_1'(t))^2+(x_2'(t))^2}-\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\right)\rho(x(\tau_j))dt.$$

Denote the terms in the last equality by $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$, respectively.

It is easy to calculate that

$$\frac{\partial\Phi\left(x,y\right)}{\partial\nu\left(y\right)}=\frac{i}{4}\left(\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}+i\,\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}\right),$$

where

$$\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}=\left(y-x,\,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^m k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)\,!\,m\,!}$$

and

$$\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}=$$

$$=\frac{2}{\pi}\left(\ln\frac{k\,\left|x-y\right|}{2}+C\right)\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}+\frac{2\,\left(y-x,\,\nu\left(y\right)\right)}{\pi\,\left|x-y\right|^2}J_0\left(k\left|x-y\right|\right)+$$

$$+\left(y-x,\,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)\,!\,m\,!}.$$

Denote by $\theta\left(\mathbf{a},\,\mathbf{b}\right)$ the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. As (see [12, p.403])

$$\left|\cos\theta\left(y-x,\,\nu\left(y\right)\right)\right|\leq M\left|x-y\right|^{\alpha}, \tag{11}$$

we have

$$\left|\frac{\partial J_0\left(k|x-y|\right)}{\partial\nu(y)}\right|\leq\left|x-y\right|\left|\cos\theta\left(y-x,\,\nu\left(y\right)\right)\right|\sum_{m=1}^{\infty}\frac{|k|^{2m}\left(diamL\right)^{2m-2}}{2^{2m-1}(m-1)!\,m!}\leq$$

$$\leq M\,\left|x-y\right|^{\alpha+1} \tag{12}$$

and

$$\left|\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}\right|\leq M\,\left(\left|x-y\right|^{\alpha+1}\left|\ln\left|x-y\right|\right|+\frac{1}{\left|x-y\right|^{1-\alpha}}+\left|x-y\right|^{\alpha+1}\right). \tag{13}$$

Consequently,

$$\left|\frac{\partial\Phi\left(x,\,y\right)}{\partial\nu\left(y\right)}\right|\leq\frac{M}{\left|x-y\right|^{1-\alpha}},\quad\forall\,x,y\in L,\quad x\neq y. \tag{14}$$

Then, taking into account the formula for the curvilinear integral, we obtain

$$\left|\delta_1^n\left(x\left(\tau_p\right)\right)\right| \leq M\ \|\rho\|_\infty \int_0^{R(n)} \frac{d\tau}{\tau^{1-\alpha}} \leq M\ \|\rho\|_\infty \left(R\left(n\right)\right)^\alpha .$$

Let $y \in L_j$ and $j \neq p$. From Lemma 2.1 and the inequality (11) it follows that

$$\left|\left(y - x\left(\tau_p\right), \nu\left(y\right)\right) - \left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(x\left(\tau_j\right)\right)\right)\right| = \left|\left(y - x\left(\tau_j\right), \nu\left(y\right)\right)\right| +$$

$$+ \left|\left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(y\right) - \nu\left(x\left(\tau_j\right)\right)\right)\right| \leq M\ \left|y - x\left(\tau_p\right)\right|\ \left(R\left(n\right)\right)^\alpha . \tag{15}$$

Then, by the inequality (6), we obtain

$$\left|\frac{\partial J_0\left(k\left|x\left(\tau_p\right) - y\right|\right)}{\partial\nu\left(y\right)} - \frac{\partial J_0\left(k\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right| \leq$$

$$\leq \left|\left(y - x\left(\tau_p\right), \nu\left(y\right)\right) - \left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(x\left(\tau_j\right)\right)\right)\right| \sum_{m=1}^{\infty} \frac{|k|^{2m}\left|x\left(\tau_p\right) - y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!} +$$

$$+ \left|\left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(x\left(\tau_j\right)\right)\right)\right| \sum_{m=1}^{\infty} \frac{|k|^{2m}\left|\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|^{2m-2} - \left|x\left(\tau_p\right) - y\right|^{2m-2}\right|}{2^{2m-1}\left(m-1\right)!\,m!} \leq$$

$$\leq M\ \left|y - x\left(\tau_p\right)\right|\ \left(R\left(n\right)\right)^\alpha \sum_{m=1}^{\infty} \frac{|k|^{2m}\left(diamL\right)^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!} +$$

$$+ M\ \left|y - x\left(\tau_p\right)\right|\ R\left(n\right) \sum_{m=1}^{\infty} \frac{|k|^{2m}\left(diamL\right)^{2m-3}}{2^{2m-1}\left(\left(m-1\right)!\right)^2} \leq M\ \left|y - x\left(\tau_p\right)\right|\ \left(R\left(n\right)\right)^\alpha . \tag{16}$$

Besides, from Lemma 1 and the inequalities (11) and (15), we have

$$\left|\frac{\left(y - x\left(\tau_p\right), \nu\left(y\right)\right)}{\left|x\left(\tau_p\right) - y\right|^2} - \frac{\left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(x\left(\tau_j\right)\right)\right)}{\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|^2}\right| \leq$$

$$\leq \left|\frac{\left(y - x\left(\tau_p\right), \nu\left(y\right)\right)\left(\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|^2 - \left|x\left(\tau_p\right) - y\right|^2\right)}{\left|x\left(\tau_p\right) - y\right|^2\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|^2}\right| +$$

$$+ \left|\frac{\left(y - x\left(\tau_p\right), \nu\left(y\right)\right) - \left(x\left(\tau_j\right) - x\left(\tau_p\right), \nu\left(x\left(\tau_j\right)\right)\right)}{\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|^2}\right| \leq$$

$$\leq M\left(\frac{R\left(n\right)}{\left|x\left(\tau_p\right) - y\right|^{2-\alpha}} + \frac{\left(R\left(n\right)\right)^\alpha}{\left|x\left(\tau_p\right) - y\right|}\right).$$

Then, taking into account the inequalities (3), (7), (12), (13), (15) and (16), it is not difficult to show that

$$\left|\frac{\partial N_0\left(k\left|x\left(\tau_p\right) - y\right|\right)}{\partial\nu\left(y\right)} - \frac{\partial N_0\left(k\left|x\left(\tau_p\right) - x\left(\tau_j\right)\right|\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right| \leq M\left(\frac{R\left(n\right)}{\left|x\left(\tau_p\right) - y\right|^{2-\alpha}} + \frac{\left(R\left(n\right)\right)^\alpha}{\left|x\left(\tau_p\right) - y\right|}\right).$$

As a result, we have

$$\left|\frac{\partial \Phi\left(x\left(\tau_p\right),\, y\right)}{\partial \nu\left(y\right)}-\frac{\partial \Phi\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| \leq M\left(\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|^{2-\alpha}}+\frac{\left(R\left(n\right)\right)^{\alpha}}{\left|x\left(\tau_p\right)-y\right|}\right).$$

Also, by the inequality

$$\left|\frac{\partial \Phi\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}-\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| \leq \frac{M}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}\, n\,!}, \tag{17}$$

we get

$$\left|\frac{\partial \Phi\left(x\left(\tau_p\right),\, y\right)}{\partial \nu\left(y\right)}-\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| \leq$$

$$\leq M\left(\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|^{2-\alpha}}+\frac{\left(R\left(n\right)\right)^{\alpha}}{\left|x\left(\tau_p\right)-y\right|}+\frac{1}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}\, n!}\right).$$

So,

$$\left|\delta_2^n\left(x\left(\tau_p\right)\right)\right| \leq$$

$$\leq M\, \left\|\rho\right\|_{\infty}\left(R\left(n\right)\int_{r(n)}^{diam\, L}\frac{d\tau}{\tau^{2-\alpha}}+\left(R\left(n\right)\right)^{\alpha}\int_{r(n)}^{diam\, L}\frac{d\tau}{\tau}+\frac{1}{n\,!}\int_{r(n)}^{diam\, L}\frac{d\tau}{\tau^{1-\alpha}}\right) \leq$$

$$\leq M\, \left\|\rho\right\|_{\infty}\left(\left(R\left(n\right)\right)^{\alpha}\, \left|\ln R\left(n\right)\right|+\frac{1}{n\,!}\right).$$

Let $y \in L_j$ and $j \neq p$. From Lemma 1 and the inequalities (14) and (17) it follows that

$$\left|\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| \leq \left|\frac{\partial \Phi\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right|+$$

$$+\left|\frac{\partial \Phi\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}-\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| \leq$$

$$\leq \frac{M}{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|^{1-\alpha}}+\frac{M}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}\, n\,!} \leq \frac{M}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}}, \quad \forall n \in \mathrm{N}. \tag{18}$$

Then we have

$$\left|\delta_3^n\left(x\left(\tau_p\right)\right)\right| \leq 2\,\omega\left(\rho, R\left(n\right)\right)\ \sum_{\substack{j=1\\ j\neq p}}^{n}\int_{L_j}\left|\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| dL_y \leq$$

$$\leq 2\,\omega\left(\rho, R\left(n\right)\right)\ \int_{L}\left|\frac{\partial \Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right| dL_y \leq$$

$$\leq M\,\omega\left(\rho, R\left(n\right)\right)\ \int_{L}\frac{dL_y}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}} \leq M\omega\ \left(\rho, R\left(n\right)\right).$$

Besides, taking into account Lemma 1 and the inequalities (10) and (18), we obtain

$$\left|\delta_4^n\left(x\left(\tau_p\right)\right)\right| \leq M\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq p}}^n \int_{t_{j-1}}^{t_j}\left|\frac{\partial\Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right| dt \leq$$

$$\leq M\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq p}}^n \int_{L_j}\left|\frac{\partial\Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right| dL_y \leq$$

$$\leq M\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha \int_{L}\left|\frac{\partial\Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right| dL_y \leq$$

$$\leq M\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha \int_{L}\frac{dL_y}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha}} \leq M\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha .$$

Finally, summing up the estimates obtained for the expressions $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$, and taking into account the relation $R\left(n\right)\sim\frac{1}{n}$, we complete the proof of Theorem 2. ◄

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