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CONDITIONS FOR THE EXISTENCE OF SMOOTH SOLUTIONS FOR A CLASS OF FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS

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Abstract. *In the paper, we consider a fourth-order operator-differential equation on the entire axis, the main part of which has a multiple characteristic, and we introduce the concept of its "smoothly" regular solvability. We find exact values of the norms of intermediate derivatives operators in a Sobolev-type space and indicate their connection with the conditions for the solvability of the equation under study. Note that the conditions found for "smoothly" regular solvability are sufficient and are imposed only on the operator coefficients of the operator-differential equation under consideration.*

Keywords: operator-differential equation, smooth solution, self-adjoint operator, Hilbert space, intermediate derivative operators

Mathematics Subject Classification (2020): 34G10, 35G05, 47A30, 47A50

1. Introduction

Let A be a self-adjoint positive-definite operator in a separable Hilbert space H. Denote by H_θ the scale of Hilbert spaces generated by the operator A, i.e.

$$H_\theta = Dom\left(A^\theta\right), \quad \theta \geq 0, \quad (x,y)_\theta = (A^\theta x, A^\theta y), \quad x,y \in Dom(A^\theta).$$

Consider a fourth-order operator-differential equation of the form

$$P\left(\frac{d}{\mathrm{dt}}\right)u(t) \equiv \left(-\frac{d}{\mathrm{dt}}+A\right)\left(\frac{d}{\mathrm{dt}}+A\right)^3 u(t)+$$

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$$+\sum_{j=1}^{3} A_j u^{(4-j)}(t) = f(t), \quad t \in \mathbb{R} = (-\infty, +\infty), \tag{1}$$

where A_j, $j = 1, 2, 3$, are linear, generally speaking, unbounded operators in H, $f(t) \in W_2^1(\mathbb{R}; H)$, $u(t) \in W_2^5(\mathbb{R}; H)$. Here by $W_2^m(\mathbb{R}; H)$, for integers $m \geq 1$, we mean the Hilbert space (see [18])

$$W_2^m(\mathbb{R}; H) = \left\{ u(t) : \frac{d^m u(t)}{dt^m} \in L_2(\mathbb{R}; H), A^m u(t) \in L_2(\mathbb{R}; H) \right\}$$

with the norm

$$\|u\|_{W_2^m(\mathbb{R};H)} = \left(\left\| \frac{d^m u}{dt^m} \right\|^2_{L_2(\mathbb{R};H)} + \|A^m u\|^2_{L_2(\mathbb{R};H)} \right)^{1/2},$$

where $L_2(\mathbb{R}; H)$ denotes the Hilbert space of vector-functions $f(t)$, defined in $\mathbb{R}$, with values in H, and for which

$$\|f\|_{L_2(\mathbb{R};H)} = \left(\int_{-\infty}^{+\infty} \|f(t)\|_H^2 \, \mathrm{dt} \right)^{1/2} < +\infty.$$

The derivatives are understood in the sense of the theory of distributions (see [18]).

Definition 1. *If the vector-function $u(t) \in W_2^5(\mathbb{R}; H)$ satisfies equation (1) for all $t \in \mathbb{R}$, then we will call it a smooth regular solution of the first order to equation (1).*

Definition 2. *If for any $f(t) \in W_2^1(\mathbb{R}; H)$ there exists a smooth regular solution of the first order to equation (1), satisfying the inequality*

$$\|u\|_{W_2^5(\mathbb{R};H)} \leq const \, \|f\|_{W_2^1(\mathbb{R};H)},$$

then equation (1) will be called "smoothly" regularly solvable.

In this paper, we find sufficient conditions imposed on the operator coefficients of equation (1), which ensure its "smoothly" regular solvability. At the same time, we find the exact values of the norms of intermediate derivatives operators in a Sobolev-type space, and indicate their connection with the conditions for the "smoothly" regular solvability of equation (1).

As is known, quite a lot of works are dedicated to the study of various issues on the solvability of operator-differential equations (see, for example, [1], [9], [13], [14], [19], [20], [24]-[26] and references there). Over the past 20 years, interest in operator-differential equations of the second and fourth orders has greatly increased (see, for example, [2], [7], [8], [10], [11], [17], [21]-[23], [28] and references there), since such equations have a specific application. But, unfortunately, the issues on solvability of operator-differential equations with a multiple characteristic have been considered relatively little (see, for example, [3]-[5], [12], [15], [16]), although from the application point of view such equations are also of interest and occur, in particular, in problems on the stability of plates made of plastic material (see, for example, [27]). It should be noted that the issue of existence of a unique solution $u(t) \in W_2^4(\mathbb{R}; H)$ for any $f(t) \in L_2(\mathbb{R}; H)$ for equation (1) was studied in [12]. A similar problem on the semi-axis was considered in [4].

2. Main Results

Throughout the work, $L(X,Y)$ is traditionally understood as the set of linear bounded operators operating from a Hilbert space X to another Hilbert space Y.

The following theorem holds.

Theorem 1. *Let the operators* $A_j \in L(H_j, H) \cap L(H_{j+1}, H_1)$, $j = 1,2,3$. *Then the operator* P, *operating as follows*

$$Pu(t) \equiv P(d/dt)u(t), u(t) \in W_2^5(\mathbb{R}; H),$$

is bounded from the space $W_2^5(\mathbb{R}; H)$ *into the space* $W_2^1(\mathbb{R}; H)$.

Proof. Let us first take the following notation:

$$P_0 u(t) \equiv P_0(d/dt)u(t) \equiv \left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^3 u(t), \quad u(t) \in W_2^5(\mathbb{R}; H),$$

$$P_1 u(t) \equiv P_1(d/dt)u(t) \equiv \sum_{j=1}^{3} A_j u^{(4-j)}(t), \quad u(t) \in W_2^5(\mathbb{R}; H).$$

For any $u(t) \in W_2^5(\mathbb{R}; H)$ the inequality holds

$$\|P_0 u\|^2_{W_2^1(\mathbb{R};H)} = \left\| -\frac{d^4u}{dt^4} - 2A\frac{d^3u}{dt^3} + 2A^3\frac{du}{dt} + A^4 u \right\|^2_{W_2^1(\mathbb{R};H)} =$$

$$= \left\| -\frac{d^5u}{dt^5} - 2A\frac{d^4u}{dt^4} + 2A^3\frac{d^2u}{dt^2} + A^4\frac{du}{dt} \right\|^2_{L_2(\mathbb{R};H)} +$$

$$+ \left\| -A\frac{d^4u}{dt^4} - 2A^2\frac{d^3u}{dt^3} + 2A^4\frac{d^2u}{dt^2} + A^5 u \right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\leq \left(\left\|\frac{d^5u}{dt^5}\right\|_{L_2(\mathbb{R};H)} + \left\|2A\frac{d^4u}{dt^4}\right\|_{L_2(\mathbb{R};H)} + \left\|2A^3\frac{d^2u}{dt^2}\right\|_{L_2(\mathbb{R};H)} + \left\|A^4\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)} \right)^2 +$$

$$+ \left(\left\|A\frac{d^4u}{dt^4}\right\|_{L_2(\mathbb{R};H)} + \left\|2A^2\frac{d^3u}{dt^3}\right\|_{L_2(\mathbb{R};H)} + \left\|2A^4\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)} + \left\|A^5 u\right\|_{L_2(\mathbb{R};H)} \right)^2.$$

Using the intermediate derivatives theorem [18]

$$\left\| A^j \frac{d^{5-j}u}{dt^{5-j}} \right\|_{L_2(\mathbb{R};H)} \leq c_j \|u\|_{W_2^5(\mathbb{R};H)}, \quad j = 0,1,2,3,4,5,$$

from the last inequality we have

$$\|P_0 u\|_{W_2^1(\mathbb{R};H)} \leq const \, \|u\|_{W_2^5(\mathbb{R};H)}. \tag{2}$$

On the other hand,

$$\|P_1u\|^2_{W_2^1(\mathbb{R};H)} = \left\|\sum_{j=1}^{3} A_j u^{(5-j)}\right\|^2_{L_2(\mathbb{R};H)} + \left\|\sum_{j=1}^{3} AA_j u^{(4-j)}\right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\leq \left(\sum_{j=1}^{3} \left\|A_j A^{-j} A^j u^{(5-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2 + \left(\sum_{j=1}^{3} \left\|AA_j A^{-(j+1)} A^{j+1} u^{(4-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2 \leq$$

$$\leq \left(\sum_{j=1}^{3} \left\|A_j \; A^{-j}\right\| \left\|A^j \; u^{(5-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2 + \left(\sum_{j=1}^{3} \left\|AA_j \; A^{-(j+1)}\right\| \left\|A^{j+1} \; u^{(4-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2.$$

Since $A_j \in L(H_j, H) \cap L(H_{j+1}, H_1)$, then the operators $A_j A^{-j}$ and $AA_j A^{-(j+1)}$, $j = 1, 2, 3$, are bounded in H. Taking into account again the theorem on intermediate derivatives [18], we obtain

$$\|P_1 u\|_{W_2^1(\mathbb{R};H)} \leq const \, \|u\|_{W_2^5(\mathbb{R};H)} \, . \tag{3}$$

It follows from inequalities (2) and (3) that

$$\|Pu\|_{W_2^1(\mathbb{R};H)} \leq \|P_0 u\|_{W_2^1(\mathbb{R};H)} + \|P_1 u\|_{W_2^1(\mathbb{R};H)} \leq const \, \|u\|_{W_2^5(\mathbb{R};H)} \, .$$

The theorem has been proven. ◀

Let us now study the solvability of the main part of equation (1).

The following theorem is true.

Theorem 2. *The equation $P_0 u(t) = f(t)$ has a unique smooth solution of the first order $u(t)$ for any $f(t) \in W_2^1(\mathbb{R}; H)$, and there is an inequality*

$$\|u\|_{W_2^5(\mathbb{R};H)} \leq const \, \|f\|_{W_2^1(\mathbb{R};H)} \, .$$

Proof. Let $f(t) \in W_2^1(\mathbb{R}; H)$, i.e. the norm is finite:

$$\left\|\frac{df}{dt}\right\|^2_{L_2(\mathbb{R};H)} + \|Af\|^2_{L_2(\mathbb{R};H)} = \|f\|^2_{W_2^1(\mathbb{R};H)} \, .$$

Then it follows from Parseval's equality that

$$\left\|i\lambda \widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} + \left\|A\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} < +\infty,$$

where $\widehat{f}(\lambda)$ is the Fourier transform of the function $f(t)$. If we denote by $\widehat{u}(\lambda)$ the Fourier transform of the function $u(t)$, then from $P_0 u(t) = f(t)$ we have

$$P_0\,(i\lambda)\,\widehat{u}(\lambda) = \widehat{f}(\lambda)$$

or

$$\widehat{u}(\lambda) = P_0^{-1}(i\lambda)\,\widehat{f}(\lambda), \quad \lambda \in \mathbb{R}, \tag{4}$$

From here we can determine that

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P_0^{-1}(i\lambda)\,\widehat{f}(\lambda) e^{i\lambda t} d\lambda.$$

Let us show that $u(t)$ is a smooth solution of the first order to the equation $P_0 u(t) = f(t)$.

Indeed, from Parseval's equality, taking into account equality (4), it follows that

$$\|u\|^2_{W_2^5(\mathbb{R};H)} = \left\| \frac{d^5 u}{dt^5} \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^5 u \right\|^2_{L_2(\mathbb{R};H)} =$$

$$= \left\| i\lambda^5 \widehat{u}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^5 \widehat{u}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} =$$

$$= \left\| i\lambda^5 P_0^{-1}(i\lambda)\,\widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^5 P_0^{-1}(i\lambda)\,\widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\leq \sup_{\lambda \in \mathbb{R}} \left\| \lambda^4 P_0^{-1}(i\lambda) \right\|^2_{H \to H} \left\| i\lambda \widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} +$$

$$+ \sup_{\lambda \in \mathbb{R}} \left\| A^4 P_0^{-1}(i\lambda) \right\|^2_{H \to H} \cdot \left\| A\widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)}. \tag{5}$$

On the other hand, the spectral expansion of the operator A implies that

$$\sup_{\lambda \in \mathbb{R}} \left\| \lambda^4 P_0^{-1}(i\lambda) \right\|_{H \to H} = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| \lambda^4 (i\lambda + \sigma)^{-3} (-i\lambda + \sigma)^{-1} \right| =$$

$$= \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| \lambda^4 (i\lambda + \sigma)^{-2} \left(\lambda^2 + \sigma^2 \right)^{-1} \right| = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \frac{\lambda^4}{\left(\lambda^2 + \sigma^2 \right)^2} \leq 1, \tag{6}$$

$$\sup_{\lambda \in \mathbb{R}} \left\| A^4 P_0^{-1}(i\lambda) \right\|_{H \to H} = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| \sigma^4 (i\lambda + \sigma)^{-3} (-i\lambda + \sigma)^{-1} \right| =$$

$$= \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \frac{\sigma^4}{\left(\lambda^2 + \sigma^2 \right)^2} \leq 1, \tag{7}$$

where $\sigma(A)$ denotes the spectrum of the operator A. Taking into account (6) and (7) in inequality (5), we obtain:

$$\|u\|^2_{W_2^5(\mathbb{R};H)} \leq \left\| i\lambda \widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} + \left\| A\widehat{f}(\lambda) \right\|^2_{L_2(\mathbb{R};H)} = \|f\|^2_{W_2^1(\mathbb{R};H)}.$$

It is evident that $u(t)$ satisfies the equation $P_0 u(t) = f(t)$. Therefore, $u(t)$ is a smooth solution of the first order to the equation $P_0 u(t) = f(t)$. The theorem has been proven.

◄

It follows from Theorem 2 that the norms $\|P_0 u\|_{W_2^1(\mathbb{R};H)}$ and $\|u\|_{W_2^5(\mathbb{R};H)}$ are equivalent in the space $W_2^5(\mathbb{R};H)$. Therefore, by the theorem on intermediate derivatives [18], the following numbers are finite:

$$n_j = \sup_{0\neq u\in W_2^5(\mathbb{R};H)} \left\|A^{4-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)} \|P_0 u\|^{-1}_{W_2^1(\mathbb{R};H)}, \quad j=1,2,3.$$

Here arises an interesting problem about the exact calculation of n_j, $j=1,2,3$. But first, let us formulate the following lemma from [16], which we use in our further reasoning.

Lemma. *Let* $\beta \in \left[0, a_j^{-1}\right)$, *where* $a_j = \frac{1}{256} j^j (4-j)^{4-j}$, $j=1,2,3$. *Then the operator pencils*

$$\widetilde{P}_j(\lambda;\beta;A) = \left(-\lambda^2 E + A^2\right) P_j(\lambda;\beta;A), \quad j=1,2,3, \tag{8}$$

where

$$P_j(\lambda;\beta;A) = \left(-\lambda^2 E + A^2\right)^4 - \beta\,(i\lambda)^{2j} A^{8-2j}, \quad j=1,2,3,$$

dependent on the parameter β *(*E *is a unit operator), are invertible on the imaginary axis and there are points* $\xi_{0,j}\in\mathbb{R}, j=1,2,3,$ *such that the characteristic polynomials*

$$\widetilde{P}_j(i\xi;\beta;\sigma) = \left(\xi^2+\sigma^2\right)\left(\left(\xi^2+\sigma^2\right)^4 - \beta\xi^{2j}\sigma^{8-2j}\right), \quad j=1,2,3, \quad \sigma\in\sigma(A),$$

satisfy the following properties:

$$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) > 0 \quad for \quad \beta\in\left[0,a_j^{-1}\right), \quad j=1,2,3, \quad \sigma\in\sigma(A);$$

$$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) = 0 \quad for \quad \beta = a_j^{-1}, \quad j=1,2,3, \quad \sigma\in\sigma(A);$$

$$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) < 0 \quad for \quad \beta > a_j^{-1}, \quad j=1,2,3, \quad \sigma\in\sigma(A).$$

Theorem 3. *The following equalities take place:*

$$n_j = a_j^{1/2}, \quad j=1,2,3.$$

Proof. Let the functions $u(t)\in W_2^5(\mathbb{R};H)$ have compact carriers and be infinitely differentiable. Then by Parseval's equality we have:

$$\|P_0 u\|^2_{W_2^1(\mathbb{R};H)} - \beta\left\|A^{4-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)} =$$

$$= \left\|-\frac{d^4u}{dt^4} - 2A\frac{d^3u}{dt^3} + 2A^3\frac{du}{dt} + A^4 u\right\|^2_{W_2^1(\mathbb{R};H)} -$$

$$-\beta\left\|A^{4-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)} = \left\|-\frac{d^5u}{dt^5} - 2A\frac{d^4u}{dt^4} + 2A^3\frac{d^2u}{dt^2} + A^4\frac{du}{dt}\right\|^2_{L_2(\mathbb{R};H)} +$$

$$+\left\|-A\frac{d^4u}{dt^4} - 2A^2\frac{d^3u}{dt^3} + 2A^4\frac{du}{dt} + A^5 u\right\|^2_{L_2(\mathbb{R};H)} -$$

$$-\beta\left(\left\|A^{4-j}u^{(j+1)}\right\|^2_{L_2(\mathbb{R};H)}+\left\|A^{5-j}u^{(j)}\right\|^2_{L_2(\mathbb{R};H)}\right)=$$

$$=\left\|\left(-(i\xi)^5E-2(i\xi)^4A+2(i\xi)^2A^3+(i\xi)A^4\right)\widehat{u}(\xi)\right\|^2_{L_2(\mathbb{R};H)}+$$

$$+\left\|\left(-(i\xi)^4A-2(i\xi)^3A^2+2(i\xi)A^4+A^5\right)\widehat{u}(\xi)\right\|^2_{L_2(\mathbb{R};H)}-$$

$$-\beta\left(\left\|(i\xi)^{j+1}A^{4-j}\widehat{u}(\xi)\right\|^2_{L_2(\mathbb{R};H)}+\left\|(i\xi)^{j}A^{5-j}\widehat{u}(\xi)\right\|^2_{L_2(\mathbb{R};H)}\right)=$$

$$=\int_{-\infty}^{+\infty}\left(\left(-(i\xi)^5E-2(i\xi)^4A+2(i\xi)^2A^3+(i\xi)A^4\right)\widehat{u}(\xi),\right.$$

$$\left.\left(-(i\xi)^5E-2(i\xi)^4A+2(i\xi)^2A^3+(i\xi)A^4\right)\widehat{u}(\xi)\right)_H d\xi+$$

$$+\int_{-\infty}^{+\infty}\left(\left(-(i\xi)^4A-2(i\xi)^3A^2+2(i\xi)A^4+A^5\right)\widehat{u}(\xi),\right.$$

$$\left.\left(-(i\xi)^4A-2(i\xi)^3A^2+2(i\xi)A^4+A^5\right)\widehat{u}(\xi)\right)_H d\xi-$$

$$-\beta\int_{-\infty}^{+\infty}\left((i\xi)^{j+1}A^{4-j}\widehat{u}(\xi),(i\xi)^{j+1}A^{4-j}\widehat{u}(\xi)\right)_H d\xi-$$

$$-\beta\int_{-\infty}^{+\infty}\left((i\xi)^{j}A^{5-j}\widehat{u}(\xi),(i\xi)^{j}A^{5-j}\widehat{u}(\xi)\right)_H d\xi=$$

$$=\int_{-\infty}^{+\infty}\left((\xi^2E+A^2)\left((\xi^2E+A^2)^4-\beta\xi^{2j}A^{8-2j}\right)\widehat{u}(\xi),\widehat{u}(\xi)\right)_H d\xi,\quad j=1,2,3.$$

Thus, for any $\beta\geq 0$ and sufficiently smooth $u(t)$, we have the equalities

$$\|P_0u\|^2_{W_2^1(\mathbb{R};H)}-\beta\left\|A^{4-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)}=$$

$$=\int_{-\infty}^{+\infty}\left(\widetilde{P}_j(i\xi;\beta;A)\widehat{u}(\xi),\widehat{u}(\xi)\right)_H d\xi,\quad j=1,2,3,\tag{9}$$

where $\widetilde{P}_j(i\xi;\beta;A)$ are defined from the equalities (8). By the Lemma, at $\beta\in\left[0,a_j^{-1}\right)$ the operator pencils $\widetilde{P}_j(i\xi;\beta;A)>0$, $j=1,2,3$. Therefore, at $\beta\in\left[0,a_j^{-1}\right)$ from (9) it follows that

$$\|P_0u\|^2_{W_2^1(\mathbb{R};H)}-\beta\left\|A^{4-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)}>0,\quad j=1,2,3.$$

Passing here to the limit as $\beta\to a_j^{-1}$, $j=1,2,3$, we have

$$\left\|A^{4-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)}\leq a_j\|P_0u\|^2_{W_2^1(\mathbb{R};H)},\quad j=1,2,3,$$

or

$$\left\| A^{4-j} u^{(j)} \right\|_{W_2^1(\mathbb{R};H)} \leq a_j^{1/2} \left\| P_0 u \right\|_{W_2^1(\mathbb{R};H)}, \quad j = 1, 2, 3,$$

i.e. $n_j \leq a_j^{1/2}$, $j = 1, 2, 3$. To prove the equalities $n_j = a_j^{1/2}$, $j = 1, 2, 3$, we define the functional

$$E(u) = \left\| P_0 u \right\|^2_{W_2^1(\mathbb{R};H)} - \beta \left\| A^{4-j} u^{(j)} \right\|^2_{W_2^1(\mathbb{R};H)}$$

in the space $W_2^5(\mathbb{R};H)$, and for $\forall \varepsilon > 0$ we find a vector-function $u_\varepsilon(t) = g_\varepsilon(t)\psi_\varepsilon$, for which $E(u_\varepsilon) < 0$, where $\psi_\varepsilon \in Dom(A^{10})$, $g_\varepsilon(t)$ is a numeric function. For this purpose, we write the inequality $E(u_\varepsilon) < 0$ in the form

$$E(u_\varepsilon) = \int_{-\infty}^{+\infty} \left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) \left| \widehat{g}_\varepsilon(\xi) \right|^2 d\xi < 0.$$

If A has at least one eigenvalue σ, then for ψ_ε we choose the corresponding eigenvector $A\psi_\varepsilon = \sigma\psi_\varepsilon$, $\|\psi_\varepsilon\| = 1$. Then it is obvious that

$$\left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right)_H = \widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; \sigma \right),$$

but by the Lemma $\xi = \xi_{0,j}$, $\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; \sigma \right) < 0$, $\forall \varepsilon > 0$. If the operator A does not have an eigenvalue, then for any $\sigma \in \sigma(A)$ and for any $\delta > 0$ we can find a vector ψ_δ ($\|\psi_\delta\| = 1$) such that for any $s > 0$

$$A^s \psi_\delta = \sigma^s \psi_\delta + o(1, \delta) \ \text{ for } \ \delta \to 0, \quad s > 0.$$

Then

$$\left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\delta, \psi_\delta \right)_H = \widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; \sigma \right) + o(1, \delta) \ \text{ for } \ \delta \to 0.$$

For sufficiently small $\delta > 0$, $\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; \sigma \right) + o(1, \delta) < 0$.

Thus, for some $\xi = \xi_{0,j}$ and $\psi_\varepsilon \in Dom\left(A^{10}\right)$, $\varepsilon > 0$

$$\left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) < 0. \tag{10}$$

Since $\left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right)$ is a continuous function of the argument ξ, the inequality (10) holds for some $\xi \in (\eta_1, \eta_2)$. Let us now build the function $g_\varepsilon(t)$ as follows. Let $\widehat{g}_\varepsilon(t)$ be infinitely differentiable function with a carrier in the interval (η_1, η_2). Denote by

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{\eta_1}^{\eta_2} \widehat{g}_\varepsilon(\xi) e^{i\xi t} d\xi$$

It's obvious that $g_\varepsilon(t) \in W_2^5(\mathbb{R})$. Then

$$E(u_\varepsilon) = E\left(g_\varepsilon(t)\psi_\varepsilon\right) = \int_{\eta_1}^{\eta_2} \left(\widetilde{P}_j \left(i\xi; a_j^{-1} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) \left| \widehat{g}_\varepsilon(\xi) \right|^2 d\xi < 0,$$

thus, it is proved that $n_j = a_j^{1/2}$, $j = 1, 2, 3$. The theorem has been proven. ◄

The results obtained allow us to establish a theorem on the "smoothly" regular solvability of equation (1) in terms of its operator coefficients.

Theorem 4. *Let A be a self-adjoint positive-definite operator in H and the operators*

$$A_j \in L\left(H_j, H\right) \cap L\left(H_{j+1}, H_1\right), \quad j = 1,2,3,$$

and the following inequality holds

$$\sum_{j=1}^{3} \max\left\{\left\|A_{4-j}A^{-(4-j)}\right\|_{H\to H}, \left\|AA_{4-j}A^{-(5-j)}\right\|_{H\to H}\right\} n_j < 1,$$

where n_j, $j = 1,2,3$, are determined from Theorem 3. Then equation (1) is "smoothly" regularly solvable.

Proof. Using the notation introduced earlier, we represent equation (1) as an operator equation

$$P_0 u(t) + P_1 u(t) = f(t), \tag{11}$$

where $f(t) \in W_2^1(\mathbb{R}; H), u(t) \in W_2^5(\mathbb{R}; H)$. By Theorem 1, the operator $P = P_0 + P_1$ is a bounded operator from the space $W_2^5(\mathbb{R}; H)$ into the space $W_2^1(\mathbb{R}; H)$, and by Theorem 2, the operator P_0 maps $W_2^5(\mathbb{R}; H)$ isomorphically onto $W_2^1(\mathbb{R}; H)$. Then there is a bounded inverse P_0^{-1}, operating from $W_2^1(\mathbb{R}; H)$ into $W_2^5(\mathbb{R}; H)$. If in equation (11) we make the substitution $u(t) = P_0^{-1}\upsilon(t)$, where $\upsilon(t) \in W_2^1(\mathbb{R}; H)$, then we obtain

$$\left(E + P_1 P_0^{-1}\right)\upsilon(t) = f(t).$$

Now let us show that under the conditions of the theorem the norm of the operator $P_1 P_0^{-1}$ is less than one. We have

$$\left\|P_1 P_0^{-1}\upsilon\right\|_{W_2^1(\mathbb{R};H)} = \|P_1 u\|_{W_2^1(\mathbb{R};H)} \le \sum_{j=1}^{3}\left\|A_j u^{(4-j)}\right\|_{W_2^1(\mathbb{R};H)} =$$

$$= \sum_{j=1}^{3}\left(\left\|A_{4-j}u^{(j+1)}\right\|^2_{L_2(\mathbb{R};H)} + \left\|AA_{4-j}u^{(j)}\right\|^2_{L_2(\mathbb{R};H)}\right)^{\frac{1}{2}} \le$$

$$\le \sum_{j=1}^{3}\left(\left\|A_{4-j}A^{-(4-j)}\right\|^2_{H\to H}\left\|A^{4-j}u^{(j+1)}\right\|^2_{L_2(\mathbb{R};H)} + \right.$$

$$\left. + \left\|AA_{4-j}A^{-(5-j)}\right\|^2_{H\to H}\left\|A^{5-j}u^{(j)}\right\|^2_{L_2(\mathbb{R};H)}\right)^{\frac{1}{2}} \le$$

$$\le \sum_{j=1}^{3}\max\left\{\left\|A_{4-j}A^{-(4-j)}\right\|_{H\to H}, \left\|AA_{4-j}A^{-(5-j)}\right\|_{H\to H}\right\} \times$$

$$\times \left(\left\|A^{4-j}u^{(j+1)}\right\|^2_{L_2(\mathbb{R};H)} + \left\|A^{5-j}u^{(j)}\right\|^2_{L_2(\mathbb{R};H)}\right)^{\frac{1}{2}}.$$

Since

$$\left\|A^{4-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)}=\left(\left\|A^{4-j}u^{(j+1)}\right\|^2_{L_2(\mathbb{R};H)}+\left\|A^{5-j}u^{(j)}\right\|^2_{L_2(\mathbb{R};H)}\right)^{\frac{1}{2}},$$

then

$$\left\|P_1P_0^{-1}v\right\|_{W_2^1(\mathbb{R};H)}\le$$

$$\le\sum_{j=1}^{3}\max\left\{\left\|A_{4-j}A^{-(4-j)}\right\|_{H\to H},\ \left\|AA_{4-j}A^{-(5-j)}\right\|_{H\to H}\right\}\left\|A^{4-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)}\le$$

$$\le\sum_{j=1}^{3}\max\left\{\left\|A_{4-j}A^{-(4-j)}\right\|_{H\to H},\left\|AA_{4-j}A^{-(5-j)}\right\|_{H\to H}\right\}n_j\left\|P_0u\right\|_{W_2^1(\mathbb{R};H)}=$$

$$=\sum_{j=1}^{3}\max\left\{\left\|A_{4-j}A^{-(4-j)}\right\|_{H\to H},\ \left\|AA_{4-j}A^{-(5-j)}\right\|_{H\to H}\right\}n_j\left\|v\right\|_{W_2^1(\mathbb{R};H)}.$$

Thus, $\left\|P_1P_0^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^1(\mathbb{R};H)}<1$. Hence, the operator $E+P_1P_0^{-1}$ has an inverse in the space $W_2^1(\mathbb{R};H)$ and, therefore, $u(t)$ can be defined by the formula

$$u(t)=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f(t),$$

at that

$$\|u\|_{W_2^5(\mathbb{R};H)}\le\left\|P_0^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^5(\mathbb{R};H)}\left\|\left(E+P_1P_0^{-1}\right)^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^1(\mathbb{R};H)}\times$$

$$\times\|f\|_{W_2^1(\mathbb{R};H)}\le const\,\|f\|_{W_2^1(\mathbb{R};H)}.$$

The theorem has been proven. ◄

It follows from Theorem 4 that the operator P is an isomorphism between the spaces $W_2^5(\mathbb{R};H)$ and $W_2^1(\mathbb{R};H)$.

Note that the main result of the paper was announced by the authors in [6].

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